

# OPEN BOOK FOLIATION II

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**ABSTRACT.** We introduce an essential open book foliation, which is a refinement of the open book foliation and establish relationship between the (essential) open book foliation and the fractional Dehn twist coefficient. As applications we characterize non-right-veering open books via the open book foliation and give sufficient conditions for a 3-manifold to be irreducible and atoroidal. We also show that the geometry of a 3-manifold is determined for some special case by the Nielsen-Thurston type of monodromy of its open book decomposition.

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## 1. INTRODUCTION

We have introduced the open book foliation in [24]. Let  $S = S_{g,r}$  be a compact oriented genus  $g$  surface with  $r \neq 0$  boundary components. Let  $\text{Aut}(S, \partial S)$  be the group of isotopy classes of diffeomorphisms of  $S$  fixing  $\partial S$  pointwise, which is isomorphic to the mapping class group  $\mathcal{MCG}(S)$  of  $S$ . By abuse of notation we will often regard  $\phi \in \text{Aut}(S, \partial S)$  as a diffeomorphism representing  $\phi$ . Suppose that a closed oriented 3-manifold  $M$  admits an open book decomposition  $(S, \phi)$ . Consider a compact oriented surface  $F$  with or without boundary embedded in  $M$ . Under certain conditions, the intersection of  $F$  and the pages of the open book yields a singular foliation on  $F$  which we call the open book foliation and is denoted by  $\mathcal{F}_{ob}(F)$ . Apparently the open book

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foliation has its origin in Bennequin's work [1] and Birman and Menasco's braid foliation [3, 4, 5, 6, 7, 8, 10, 11], where the underlying manifold  $M \cong S^3$ .

Let us recall Giroux's seminal result [18]: For a closed oriented 3-manifold  $M$  there exists a one-to-one correspondence between the open book decompositions of  $M$  up to positive stabilization and the contact structures on  $M$  up to contact isotopy. Due to the Giroux-correspondence the open book foliation has extensive application to contact geometry. For instance, in [24] we have reproved Honda, Kazez and Matić [20]'s theorem: The contact structure  $(M, \xi)$  is tight if and only if every open book supporting  $(M, \xi)$  is right-veering.

Now the notion "*right-veering*" is a key to tight contact structures. For every boundary component  $C \subset \partial S$  and every essential arc  $\gamma \subset S$  starting on  $C$  if  $\phi(\gamma)$  lies on the right of  $\gamma$  near the starting point, we say that  $\phi$  is right-veering. As a highly effective tool to detect right-veering-ness Honda, Kazez and Matić [20] invent the *fractional Dehn twist coefficient*. It measures how much  $\phi \in \text{Aut}(S, \partial S)$  contributes twisting along a boundary component  $C$  and is denoted by  $c(\phi, C)$ . They show that positivity of  $c(\phi, C)$  is (roughly) equivalent to the right-veering-ness of  $\phi$ .

In the present paper we introduce the (*strongly*) *essential* open book foliation and establish relationship between the fractional Dehn twist coefficient and the essential open book foliation. As applications we obtain results in contact 3-manifolds and topology and geometry of 3-manifolds.

In the previous paper [24] we start from an open book  $(S, \phi)$  with certain properties (eg. non-right-veeringness) and construct surfaces  $F$  in  $M_{(S, \phi)}$  (eg. transverse overtwisted discs and a Seifert surface of a closed braid). Combinatorial nature of the open book foliation  $\mathcal{F}_{ob}(F)$  enables us to extract various properties of the underlying contact manifold  $(M_{(S, \phi)}, \xi_{(S, \phi)})$  or the closed braid  $\partial F$ , such as overtwistedness or the self-linking number.

In this paper we explore the converse: We start from a surface admitting an open book foliation  $\mathcal{F}_{ob}(F)$  and read properties of the monodromy  $\phi$ , typically the fractional Dehn twist coefficient  $c(\phi, C)$ . For this purpose, a general open book foliation often contains excessive information. We define the (*strongly*) *essential* open book foliation as an optimal foliation for sharper estimate of  $c(\phi, C)$ . It is similar to using incompressible surfaces rather than generic ones to study topology/geometry of the ambient 3-manifolds.

In short our strategy breaks into three steps:

- (1) Replace a given open book foliation  $\mathcal{F}_{ob}$  with another one  $\mathcal{F}'_{ob}$  that reflects more properties of the monodromy  $\phi \in \text{Aut}(S, \partial S)$ .
- (2) Read the properties of  $\phi$  from  $\mathcal{F}'_{ob}$ .
- (3) Find application to contact geometry and topology/geometry of 3-manifolds.

In this paper we mainly treat Steps (2) and (3). We discuss Step (2) in Section 5. Sections 6–8 serve for Step (3).

As for Step (1), typically we think that  $\mathcal{F}_{ob}$  is a generic open book foliation and  $\mathcal{F}'_{ob}$  is an essential one. Step (1) is the most difficult and delicate step. In fact, in some cases even an essential open book foliation is insufficient for our purposes. To deal with the problem we further introduce *strongly* essential b-arcs and elliptic points. In the subsequent paper [25], we will study more advanced techniques (some foliation moves and braid moves) to convert a given  $\mathcal{F}_{ob}$  to a more useful  $\mathcal{F}'_{ob}$ .

The paper is organized as follows:

In Section 2 we review basic notions and facts about the open book foliation.

In Section 3 we introduce the essential open book foliation, and show that every incompressible surface can be isotoped so that it admits an essential open book foliation (Theorem 3.2). We introduce a notion of *strongly essential* b-arc which plays a crucial role in large part of the paper.

In Section 4 we develop basics of the fractional Dehn twist coefficient  $c(\phi, C)$ . We give a practical and efficient method to compute  $c(\phi, C)$  which does not require Nielsen-Thurston classification (Theorem 4.7), and interpret  $c(\phi, C)$  as a translation number of certain dynamics (Theorem 4.9).

Section 5 is the core of the paper. In Theorem 5.3 we obtain lower and upper bounds of  $c(\phi, C)$  by counting the number of singularities of a strongly essential open book foliation.

In Section 6 we apply Theorem 5.3 to characterize non-right-veering monodromy: We show that a monodromy is non-right-veering if and only if there exists a special “simplest” transverse overtwisted disc (Theorem 6.2). This highlights the difference between non-right-veering monodromy and overtwistedness:  $\phi$  being right-veering does *not* always imply that  $\xi_{(S, \phi)}$  is overtwisted.

In Section 7 we apply our study to topology of 3-manifolds. We obtain bounds of  $|c(\phi, C)|$  from a closed incompressible surface embedded in  $M_{(S, \phi)}$  (Theorem 7.3 and Theorem 7.4). This relates topology of 3-manifolds and  $c(\phi, C)$ . As a consequence, we find sufficient conditions on the monodromy  $\phi$  that  $M_{(S, \phi)}$  is irreducible (Theorem 7.1), or is atoroidal (Theorem 7.2).

In Section 8 we study geometry of 3-manifolds. We prove that if an open book  $(S, \phi)$  with connected binding has  $|c(\phi, \partial)| > 1$  then the Nielsen-Thurston type of  $\phi$  determines the geometric structure of  $M_{(S, \phi)}$  and vice versa (Theorem 8.5). Our result is parallel to Thurston’s classification of geometry of mapping tori.

## 2. QUICK REVIEW OF OPEN BOOK FOLIATION

In this section we review basic definitions and techniques of the open book foliation. For details see [24, §2-4].

An *open book*  $(S, \phi)$  is a pair of oriented compact surface  $S$  with non-empty boundary  $\partial S$  and (the isotopy class of) a diffeomorphism  $\phi \in \text{Aut}(S, \partial S)$  fixing the boundary pointwise. By abuse of notation we will often use  $\phi$  for its diffeomorphism representative. Given an open book  $(S, \phi)$  we define a closed oriented 3-manifold  $M_{(S, \phi)}$  by

$$M_{(S, \phi)} = M_\phi \cup \left( \coprod_{|\partial S|} D^2 \times S^1 \right)$$

where  $M_\phi$  denotes the mapping torus  $S \times [0, 1]/(x, 1) \sim (\phi(x), 0)$ , and the solid tori are attached so that for each point  $p \in \partial S$  the circle  $\{p\} \times S^1 \subset \partial M_\phi$  bounds a meridian disc of  $D^2 \times S^1$ . We say that  $(S, \phi)$  is an *open book decomposition* of the 3-manifold  $M = M_{(S, \phi)}$ . We view the cores of the attached solid tori as an oriented fibered link in  $M$  and call it the *binding*  $B$  of the open book. Let  $\pi : M \setminus B \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$  denote the fibration. The fibers  $\pi^{-1}(t) = S_t$  where  $t \in [0, 1)$  are called the *pages* of the open book.

Let  $\xi = \xi_{(S,\phi)} = \ker \alpha$  be the contact structure on  $M$  supported by  $(S, \phi)$  through the Giroux-correspondence: That is,  $\alpha > 0$  on the binding  $B$  and  $d\alpha$  is a positive area form on each page  $S_t$ . See [30] for a construction of such contact structure.

We say that an oriented link  $L$  in  $M$  is in *braid position* with respect to  $(S, \phi)$  if  $L$  is disjoint from  $B$  and positively transverse to each page  $S_t$ . The algebraic intersection number of  $L$  and the page  $S_0$  is called the *braid index* of  $L$ . Thanks to Bennequin [1] and Pavelescu [28], any transverse link in a contact 3-manifold  $(M, \xi)_{(S,\phi)}$  can be transversely isotoped to a closed braid in  $(S, \phi)$ . Conversely, a closed braid in  $(S, \phi)$  is naturally regarded as a transverse link in  $(M, \xi)$ . Hence from now on, we always assume that every (transverse) link is in braid position.

Fix a closed (possibly empty) braid  $L$ . Let  $F$  be an oriented connected compact surface embedded in  $M$ . We consider one of the following situations:

- $F$  is a closed surface that lies in  $M \setminus L$ .
- $F$  is a Seifert surface of  $L$ , i.e.,  $\partial F = L$ .

Consider the singular foliation  $\mathcal{F} = \mathcal{F}(F)$  of  $F$  induced by the intersections of fibers  $\{S_t\}$  and  $F$ . We call each connected component of  $F \cap S_t$  a *leaf*.

**Definition 2.1.** We say that the above  $\mathcal{F}$  is an *open book foliation*, denoted by  $\mathcal{F}_{ob}(F)$ , if the following four conditions are satisfied.

- (OF i): The binding  $B$  pierces the surface  $F$  transversely in finitely many points. Each point  $p \in B \cap F$  is an *elliptic* singularity of  $\mathcal{F}$ . See [17, p.166] for the definition of elliptic points. Geometrically, there exists a disc neighborhood  $N_p \subset F$  of  $p$  on which the foliation  $\mathcal{F}(N_p)$  is radial with the node  $p$ . The converse also holds: any elliptic singularity of  $\mathcal{F}$  is a transverse intersection point of  $B$  and  $\text{Int}(F)$ .
- (OF ii): There exists a tubular neighborhood  $N(L) \subset M$  of  $L$  such that each leaf of the foliation  $\mathcal{F}(F \cap N(L))$  transversely intersects  $L$ .
- (OF iii): All but finitely many fibers  $S_t$  intersect  $F$  transversely. Each exceptional fiber is tangent to  $\text{Int}(F)$  at a single point. In particular, saddle-saddle connections do not exist.
- (OF iv): All the tangencies of  $F$  with fibers are saddles, and each saddle corresponds to a hyperbolic singularity of  $\mathcal{F}(F)$ .

We say that a page  $S_t$  is *regular* if  $S_t$  intersects  $F$  transversely and is *singular* otherwise. Similarly, we say a leaf  $l$  of  $\mathcal{F}$  is *regular* if  $l$  does not contain a tangency point and is *singular* otherwise. The regular leaves are classified into the following three types:

- a-arc : An arc where one of its endpoints lies on  $B$  and the other lies on  $L$ .
- b-arc : An arc whose endpoints both lie on  $B$ .
- c-circle : A simple closed curve.

**Theorem 2.2.** [24, Theorem 3.5] *By isotopy that fixes the transverse link type of the boundary  $\partial F$  (if  $\partial F$  exists), every surface  $F$  admits an open book foliation  $\mathcal{F}_{ob}(F)$ . Moreover, we may also assume that  $\mathcal{F}_{ob}(F)$  has no c-circle leaves.*

The open book foliation has two kinds of singularities: An *elliptic* point which is a point of  $B \cap F$ , and a *hyperbolic* point where  $F$  is tangent to a fiber  $S_t$ . We say that an elliptic point  $p$  is *positive* (resp. *negative*) if the binding  $B$  is positively (resp. negatively)

transverse to  $F$  at  $p$ . The sign of the hyperbolic point  $q$  is *positive* (resp. *negative*) if the positive normal direction of  $F$  at  $q$  agrees (resp. disagrees) with the direction of  $t$ . See Figure 1, where we describe an elliptic point by a hollow circle with its sign inside, a hyperbolic point by a dot with the sign nearby, and positive normals to  $F$ ,  $\vec{n}_F$ , by dashed arrows.

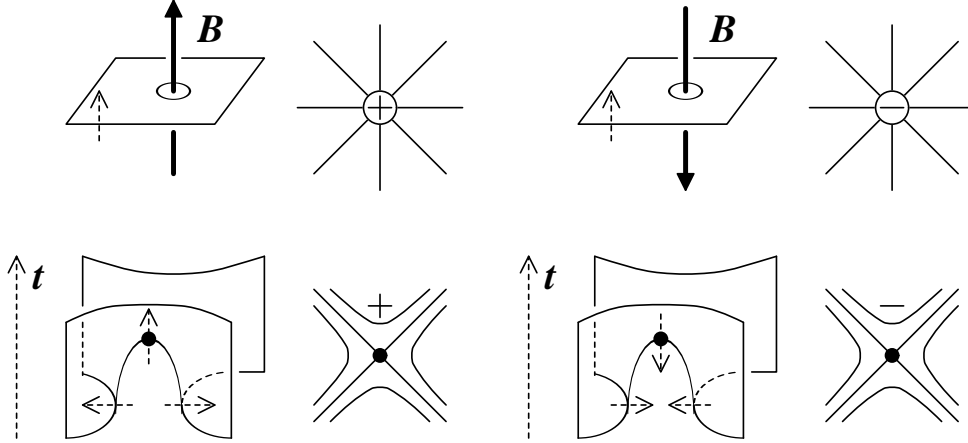


FIGURE 1. Signs of the singularities in  $\mathcal{F}_{ob}$  and normal vectors  $\vec{n}_F$ .

We see a hyperbolic singularity as a process of switching the configuration of leaves. As  $t$  increases, two regular leaves  $l_1$  and  $l_2$  approach along an arc  $\gamma$  (the dashed arc in Figure 2) connecting  $l_1$  and  $l_2$ . At a critical moment  $l_1$  and  $l_2$  form a hyperbolic singularity, then the configuration is changed. See the passage in Figure 2. The hyperbolic

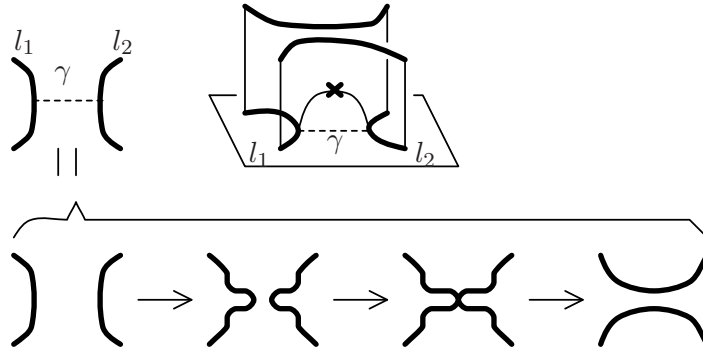


FIGURE 2. A description arc (dashed) for a hyperbolic singularity.

singularity is determined by the isotopy class of  $\gamma$ . We call  $\gamma$  a *description arc* of the hyperbolic singularity and use a dashed arc. We will denote the sign of a singular point  $x$  of  $\mathcal{F}_{ob}$  by  $\text{sgn}(x)$ .

**Definition 2.3.** We denote the number of positive (resp. negative) elliptic points of  $\mathcal{F}_{ob}(F)$  by  $e_+ = e_+(\mathcal{F}_{ob}(F))$  (resp.  $e_- = e_-(\mathcal{F}_{ob}(F))$ ). Similarly, the number of positive (resp. negative) hyperbolic points is denoted by  $h_+ = h_+(\mathcal{F}_{ob}(F))$  (resp.  $h_- = h_-(\mathcal{F}_{ob}(F))$ ).

Recall that the *characteristic foliation*  $\mathcal{F}_\xi(F)$  of an embedded surface  $F \subset (M, \xi)$  is a singular foliation obtained by integrating the vector field  $\xi \cap TF$  on  $F$ . The characteristic foliation and the convex surfaces play important roles in contact geometry. The next theorem shows a close relation between  $\mathcal{F}_{ob}(F)$  and  $\mathcal{F}_\xi(F)$ . In particular, an open book foliation without c-circles is regarded as a characteristic foliation. For detailed comparison of  $\mathcal{F}_{ob}(F)$  and  $\mathcal{F}_\xi(F)$ , see [24, Remark 4.2].

**Theorem 2.4** (Structural stability theorem). [24, Theorem 4.1] *Assume that a surface  $F$  in  $(S, \phi)$  admits the open book foliation  $\mathcal{F}_{ob}(F)$ . Then there exists a surface  $F'$  a  $C^1$ -small perturbation of  $F$  fixing the boundary  $\partial F$  pointwise such that  $e_\pm(\mathcal{F}_{ob}(F')) = e_\pm(\mathcal{F}_\xi(F'))$  and  $h_\pm(\mathcal{F}_{ob}(F')) = h_\pm(\mathcal{F}_\xi(F'))$ .*

*Moreover, if  $\mathcal{F}_{ob}(F)$  contains no c-circles, then we may choose  $F'$  so that  $\mathcal{F}_{ob}(F')$  and  $\mathcal{F}_\xi(F')$  are topologically conjugate, namely there exists a foliation preserving homeomorphism of  $F'$ , and that  $F'$  is a convex surface.*

The above result yields the following:

**Proposition 2.5.** [24, Propositions 5.1, 3.9] *Suppose that  $F \subset M_{(S, \phi)}$  is a surface admitting an open book foliation.*

- (1) *If  $\partial F$  is non-empty, the self linking number  $sl(\partial F, [F]) = -\langle e(\xi), [F] \rangle = -(e_+ - e_-) + (h_+ - h_-)$ .*
- (2) *The Euler characteristic  $\chi(F) = (e_+ + e_-) - (h_+ + h_-)$ .*

Hyperbolic singularities in  $\mathcal{F}_{ob}(F)$  are classified into six types, according to the types of nearby regular leaves: Type *aa*, *ab*, *bb*, *ac*, *bc*, and *cc* as depicted in Figure 3. Such a model neighborhood is called a *region neighborhood*. We denote by  $\text{sgn}(R)$  the sign of the hyperbolic point that region  $R$  contains. If  $R$  is of type *aa*, *ac*, *bc*, or *cc*, some parts of  $\partial R$  may be identified. In such case we say that  $R$  is *degenerated*.

**Proposition 2.6** (Region decomposition). [24, Proposition 3.11] *If  $\mathcal{F}_{ob}(F)$  contains a hyperbolic point, the surface  $F$  is decomposed into a union of regions so that whose interiors are disjoint. For each  $\mathcal{F}_{ob}(F)$  the decomposition is unique up to three types of regions foliated only by regular a-arcs, b-arcs and c-circles, cf. [24, Figure 9].*

We call a decomposition in Proposition 2.6 a *region decomposition* of  $F$ . If  $\mathcal{F}_{ob}(F)$  contains no c-circle leaves, then all regions are 2-cells and the region decomposition yields a cellular decomposition of  $F$ .

**Definition 2.7.** The *negativity graph*  $G_{--}$  is a graph properly embedded in  $F$ . The edges of  $G_{--}$  are the unstable separatrices for negative hyperbolic points in *aa*-, *ab*- and *bb*-tiles. See Figure 4. We regard the negative hyperbolic points as part of the edges, i.e., not vertices. The vertices of  $G_{--}$  are the negative elliptic points in *ab*- and *bb*-tiles and the end points of the edges of  $G_{--}$  that lie on  $\partial F$ , called the *fake vertices*. Similarly we can define the *positivity graph*  $G_{++}$ .

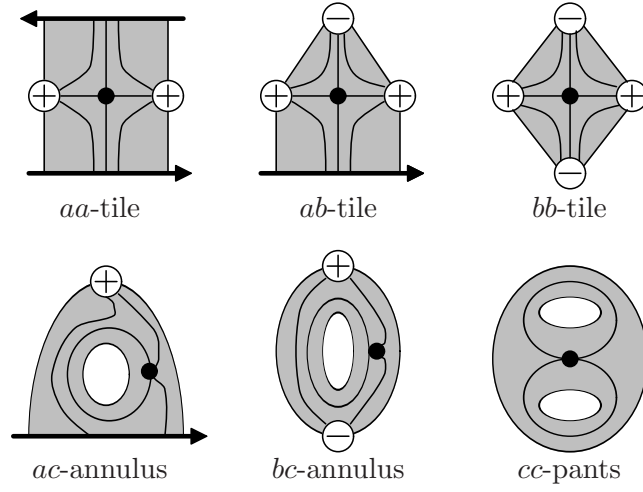
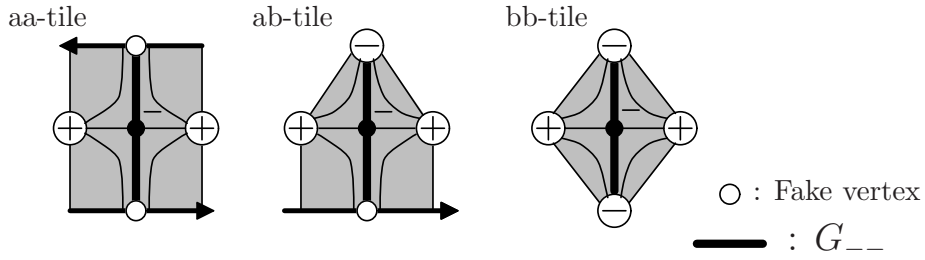


FIGURE 3. Six types of the region neighborhoods for hyperbolic singularities.


 FIGURE 4. The negativity graph  $G_{--}$ .

Finally, we recall the transverse overtwisted disc.

**Definition 2.8.** Let  $D \subset M_{(S,\phi)}$  be an embedded oriented disc whose boundary is a positive unknot braid in  $(S, \phi)$ . If the following are satisfied  $D$  is called a *transverse overtwisted disc*:

- (1)  $G_{--}$  is a tree with no fake vertices,
- (2)  $\mathcal{F}_{ob}(D)$  contains no c-circles,
- (3)  $G_{++}$  is homeomorphic to  $S^1$ .

**Theorem 2.9.** [24, Proposition 6.2 and Corollary 6.5] *The contact structure  $\xi_{(S,\phi)}$  is overtwisted if and only if there exists a transverse overtwisted disc.*

**Remark.** Definition 2.8 implies that the region decomposition of a transverse overtwisted disc consists of  $ab$ - and  $bb$ -tiles with  $\text{sgn}(ab\text{-tile}) = +1$  and  $\text{sgn}(bb\text{-tile}) = -1$ . Hence  $G_{++}$  lives in the  $ab$ -tiles and  $G_{--}$  lives in the  $bb$ -tiles.



## 3. ESSENTIAL OPEN BOOK FOLIATION

In this section we introduce an *essential* open book foliation. Take a closed braid  $L$  and a surface  $F$  as in the previous section so that  $F$  admits an open book foliation.

**Definition 3.1.** We say that a  $b$ -arc  $l$  in a fiber  $S_t$  is *essential* if  $l$  is an essential arc in  $S_t \setminus (S_t \cap L)$ . Similarly, a  $c$ -circle  $l$  in  $S_t$  is called *essential* if  $l$  is an essential simple closed curve in  $S_t \setminus (S_t \cap L)$ . The open book foliation  $\mathcal{F}_{ob}(F)$  is called *essential* if all the  $b$ -arcs are essential ( $c$ -circles need not be essential).

Our first result is an improvement of Theorem 2.2 for *incompressible* surfaces.

**Theorem 3.2.** *Assume that a surface  $F$  is incompressible. Then applying isotopy that fixes  $L, F$  admits an essential open book foliation.*

Theorem 3.2 essentially follows from Roussarie-Thurston's theorem [29, 31, 13] for a incompressible surface in a taut foliation. A codimension one foliation  $\mathfrak{F}$  on a 3-manifold is called *taut* if there exists an embedded circle which transversely intersects all the leaves of  $\mathfrak{F}$ . Roussarie-Thurston's theorem implies that an incompressible surface in a taut foliation can be put in good position similar to the setting of the open book foliation.

**Theorem 3.3** (Roussarie-Thurston [29, 31, 13]). *let  $\mathfrak{F}$  be a transversely oriented taut foliation on a 3-manifold  $M$  other than the product foliation of  $S^2 \times S^1$ . Let  $F$  be an incompressible surface in  $M$  such that  $\partial F$  transverse to  $\mathfrak{F}$  and that  $F$  is not isotopic to the leaves of  $\mathfrak{F}$ . Then by isotopy that fixes  $\partial F$ ,  $F$  can be transverse to  $\mathfrak{F}$  except for finitely many isolated saddle tangencies.*

Now we prove Theorem 3.2 by using Roussarie-Thurston theorem.

*Proof of Theorem 3.2.* We put the surface  $F$  in a general position so that it admits a singular foliation  $\mathcal{F}$  satisfying the properties **(OF i)**, **(OF ii)**, **(OF iii)** in Definition 2.1 and **(OF iv')**: The type of each tangency in **(OF iii)** is either saddle or local extremum. Note that **(OF iv')** is weaker than **(OF iv)**.

We remove all the inessential  $b$ -arcs by the following way (cf. [2, Lemma 1.2]). Let  $l$  be an innermost inessential  $b$ -arc of  $\mathcal{F}$  in a regular page  $S_t$  that cobounds with a sub-arc of the binding  $B = \partial S_t$  a disc  $\Delta \subset S_t$ , and  $\Delta \cap L$  is empty. Since  $l$  is innermost  $\Delta$  contains no  $b$ -arcs. If  $\Delta$  contains  $c$ -circles  $c_1, \dots, c_k$ , let  $A_i \subset F$  be a small annular neighborhood of  $c_i$  with no singularities. We push each annulus  $A_i$  out of  $\Delta$  across the binding  $B$  as shown in Figure 5. This does not create new  $b$ -arcs but new local extrema appear. Now  $l$  is boundary parallel in  $S_t \setminus (S_t \cap F)$ . We push the surface  $F$  along  $\Delta$  as shown in Figure 6. As a consequence the foliation  $\mathcal{F}$  changes. Namely, two elliptic points that are the end points of  $l$  disappear and new hyperbolic points and local extremal points appear. Since the number of elliptic points in the original  $\mathcal{F}$  is finite, applying the operation finitely many times  $\mathcal{F}$  no longer has inessential  $b$ -arcs.

Let  $N(B)$  be a standard tubular neighborhood of the binding  $B$ , and  $n$  be the braid index of  $L$ . Since the complement  $M \setminus (N(B) \cup L)$  is an  $(S_{g,r+n})$ -bundle over  $S^1$  it admits a taut foliation,  $\mathfrak{F}$ . By assumption,  $F' = F \setminus (F \cap N(B))$  is not isotopic to the leaves of  $\mathfrak{F}$ . By Theorem 3.3 we can isotope  $F'$  fixing  $\partial F'$  and  $L$  so that  $F'$  is transverse to  $\mathfrak{F}$  except for finitely many isolated saddle tangencies and has no local extrema. Hence the surface  $F' \cup (F \cap N(B))$  admits an open book foliation with no inessential  $b$ -arcs.  $\square$



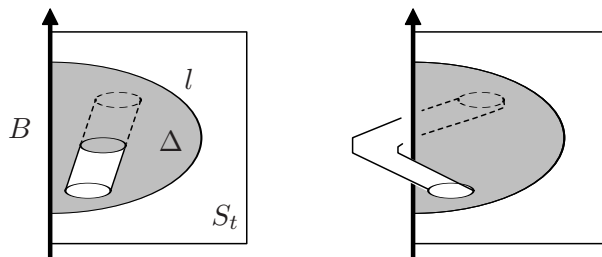
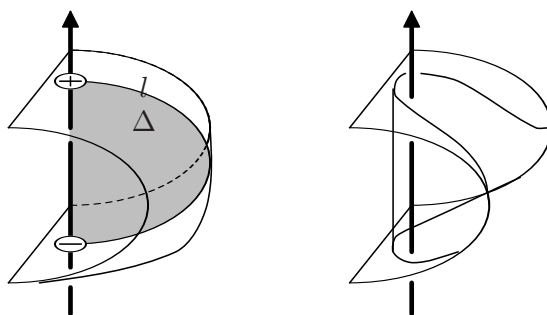
FIGURE 5. Removing c-circles in  $\Delta$ .

FIGURE 6. Removing an inessential b-arc.

**Remark.** The Roussarie-Thurston theorem cannot apply to compressible surfaces in general, hence a compressible surface may not admit essential open book foliation. The most trivial example is a compressible sphere in the complement of the binding.

In contrast to Theorem 2.2, there may exist a surface  $F$  all of whose essential open book foliations have c-circles.

The converse of Theorem 3.2 is false: a compressible surface might admit an essential open book foliation. For example, an embedded torus which is the boundary of a regular neighborhood of closed braid admits an essential open book foliation.

Next we introduce “strong essentiality”, a key concept for estimates of the fractional Dehn twist coefficient.

**Definition 3.4.** A  $b$ -arc in a fiber  $S_t$  is called *strongly essential* if it is essential in  $S_t$ . (Hence if a  $b$ -arc is strongly essential then it is essential.) An elliptic point  $v \in \mathcal{F}_{ob}(F)$  is called *strongly essential* if every  $b$ -arc that ends at  $v$  is strongly essential.

In braid foliation theory, no  $b$ -arcs are strongly essential because  $S_t \cong D^2$ . The existence of strongly essential  $b$ -arcs is a unique feature of the open book foliation.

## 4. FRACTIONAL DEHN TWIST COEFFICIENTS

In this section we review the notion of right-veering diffeomorphisms and the fractional Dehn twist coefficient in [20]. We show that the fractional Dehn twist coefficient is effectively computable and give an alternative description which does not require the Nielsen-Thurston classification.

Let  $C$  be a boundary component of  $S$ , and let  $\gamma, \gamma'$  be isotopy classes (rel. to the endpoints) of oriented properly embedded arcs in  $S$  which start at the same base point  $*$   $\in C \subset \partial S$ . We say that  $\gamma'$  lies *strictly on the right side* of  $\gamma$  if there exist curves representing  $\gamma$  and  $\gamma'$  with the minimal geometric intersection such that around the base point  $*$ ,  $\gamma'$  strictly lies on the right side of  $\gamma$ . In such case, we denote  $\gamma > \gamma'$ .

**Definition 4.1.** [20, Definition 2.1] Let  $C$  be a boundary component of  $S$ . We say that  $\phi \in \text{Aut}(S, \partial S)$  is *right-veering* (resp. *strictly right-veering*) with respect to  $C$  if  $\gamma \geq \phi(\gamma)$  (resp.  $\gamma > \phi(\gamma)$ ) for any isotopy classes  $\gamma$  of essential arcs in  $S$  starting at  $C$ . We say that  $\phi$  is *(strictly) right-veering* if  $\phi$  is (strictly) right-veering with respect to all the boundary components of  $S$ . In particular, the identity map is right-veering.

**Convention 4.2.** Assume that  $\chi(S) < 0$ , i.e.,  $S$  admits a complete hyperbolic metric with finite area. By the Nielsen-Thurston classification [12]  $\phi \in \text{Aut}(S, \partial S)$  is freely isotopic to a homeomorphism of  $S$  of type either periodic, reducible or pseudo-Anosov. For each case, we say that  $\phi \in \text{Aut}(S, \partial S)$  is *periodic*, *reducible*, *pseudo-Anosov*, respectively.

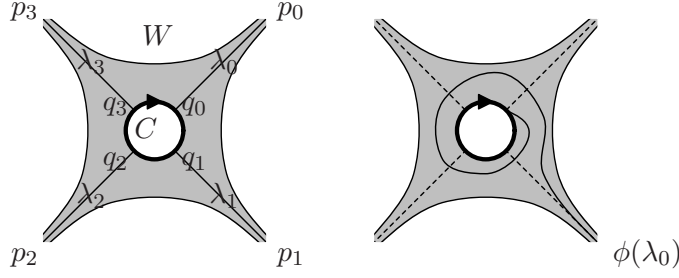
If  $S$  is an annulus or a disc, we will regard every element  $\phi \in \text{Aut}(S, \partial S)$  *periodic*.

The *fractional Dehn twist coefficient*  $c(\phi, C)$  expresses how much  $\phi$  twists the surface  $S$  around the boundary component  $C$ . This number is closely related to the right-veeringness. Let us recall the definition of  $c(\phi, C)$ .

**Definition 4.3.** [20, p.423]

(1) Assume that  $\phi \in \text{Aut}(S, \partial S)$  is periodic. Let  $C_1, \dots, C_n$  be the boundary components of  $S$ . Let  $T_C$  denote the right-handed Dehn twist along (a curve parallel to)  $C$ . There exist numbers  $N \in \mathbb{N}$  and  $M_1, \dots, M_n \in \mathbb{Z}$  such that  $\phi^N = T_{C_1}^{M_1} \dots T_{C_n}^{M_n}$ . We define the fractional Dehn twist coefficient  $c(\phi, C_i) = \frac{M_i}{N}$ , which is independent of the choice of  $N, M_1, \dots, M_n$ .

(2) Assume that  $\phi \in \text{Aut}(S, \partial S)$  is pseudo-Anosov and  $\phi$  is freely isotopic to a pseudo-Anosov homeomorphism,  $\Phi$ . Fix a boundary component  $C$ . Let  $L$  be the stable (or unstable) geodesic measured lamination for  $\Phi$  and  $W$  be the connected component of  $S \setminus L$  containing  $C$ . It is known that  $W$  is a *crown* [12, Lemma 4.4]: a complete hyperbolic surface of finite area and homeomorphic to  $(S^1 \times [0, 1]) \setminus A$  where  $C$  is identified with  $S^1 \times \{0\}$  and  $A = \{p_0, \dots, p_{m-1}\}$  a finite set of points in  $S^1 \times \{1\}$ . Take  $m$  semi-infinite geodesics  $\lambda_i \subset W$  which start at a point  $q_i \in C$  and approach to  $p_i$  (see Figure 7 left). There exists  $j \in \{0, \dots, m-1\}$  such that  $\Phi(\lambda_i) = \lambda_{i+j}$ . Let  $H : C \times [0, 1] \rightarrow C$  be the free isotopy from  $\phi$  to  $\Phi$  restricted to  $C$ . We have  $H(q_i, 0) = \phi(q_i) = q_i$  and  $H(q_i, 1) = \Phi(q_i) = q_{i+j}$ . There exists  $N \in \mathbb{Z}$  such that the arc  $\alpha_t = H(q_0, t)$  starts from  $q_0$  and winds around  $C$  in the positive direction for  $N + \frac{j}{m}$  times and then stops at  $q_j$ . We define  $c(\phi, C) = N + \frac{j}{m}$ . For example,  $\phi$  depicted in Figure 7 has  $c(\phi, C) = 1 + \frac{1}{4} = \frac{5}{4}$ .

FIGURE 7. Pseudo-Anosov case.  $c(\phi, C) = 5/4$ .

(3) Suppose that  $\phi \in \text{Aut}(S, \partial S)$  is reducible. For each boundary component  $C$  of  $S$  there exists subsurface  $S'$  of  $S$  containing  $C$  such that  $\phi(S') = S'$  and  $\phi|_{S'}$  is either periodic or pseudo-Anosov. We define  $c(\phi, C) = c(\phi|_{S'}, C)$ .

**4.1. Estimates of  $c(\phi, C)$ .** We develop computational techniques for  $c(\phi, C)$ , including the *key lemma* (Lemma 4.6). In later sections we will obtain more estimates by using the open book foliation. We start by listing basic properties of  $c(\phi, C)$  proven in [20].

**Proposition 4.4.** [20] *Let  $C$  be a boundary component of  $S$  and  $\phi \in \text{Aut}(S, \partial S)$ .*

- (1)  $c(\phi^N, C) = Nc(\phi, C)$  for  $N \in \mathbb{Z}$ .
- (2)  $c(T_C, C) = 1$  and  $c(T_C\phi, C) = 1 + c(\phi, C)$ .
- (3) *If  $\phi$  is periodic then  $\phi$  is right-veering if and only if  $c(\phi, C) \geq 0$  for all  $C$ . If  $\phi$  is pseudo-Anosov then  $\phi$  is right-veering with respect to  $C$  if and only if  $c(\phi, C) > 0$ .*

The following shows that topology of  $S$  governs  $c(\phi, C)$ .

**Proposition 4.5.** *Let  $S = S_{g,d}$  be an oriented genus  $g$  surface with  $d > 0$  boundaries. Then for  $\phi \in \text{Aut}(S, \partial S)$ ,*

$$(4.1) \quad c(\phi, C) \in \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \{1, 2, \dots, 4g + 2\} \right\}.$$

Moreover, if  $\phi$  is pseudo-Anosov, then

$$(4.2) \quad c(\phi, C) \in \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \{1, 2, \dots, 4g + d - 3\} \right\}.$$

*Proof.* If  $S$  is an annulus (or a disc), then  $c \in \mathbb{Z}$  (or  $c = 0$ ) so the statement holds. Hence in the rest of the proof we assume  $\chi(S) < 0$ . By fixing a hyperbolic metric on  $S$ , we regard  $S$  as a complete hyperbolic surface with geodesic boundary and finite area.

Assume that  $\phi$  is periodic of period  $M$ . Let  $\hat{S}$  be the genus  $g$  surface obtained by capping off the  $d$  boundary circles. Extend  $\phi$  to  $\hat{\phi} \in \text{Aut}(\hat{S})$  by setting  $\hat{\phi} = id$  on  $\hat{S} - S$ . Clearly  $\hat{\phi}$  has the period  $M$ . The “ $4g + 2$  theorem” [16, Theorem 7.5] implies that  $M \leq 4g + 2$ . Since  $Mc(\phi, C) \in \mathbb{Z}$  we get (4.1).

Next assume that  $\phi$  is pseudo-Anosov. By Definition 4.3 we have:

$$c(\phi, C) \in \left\{ \frac{p}{m} \mid p \in \mathbb{Z} \right\}.$$

Let  $L, W$  and  $m$  be as in Definition 4.3. Since  $\text{Area}(S - L) = \text{Area}(S)$  by [12, Theorem 4.9],  $\text{Area}(W) = m\pi$ , and  $\text{Area}(S - W) \geq (d - 1)\pi$  we have an inequality:

$$m\pi + (d - 1)\pi \leq \text{Area}(W) + \text{Area}(S - W) = \text{Area}(S) = (4g - 4 + 2d)\pi,$$

i.e.,  $m \leq 4g - 3 + d$  and we obtain (4.2).

When  $\phi$  is reducible the statement follows from the above argument.  $\square$

The following is a key estimate of  $c(\phi, C)$  that we will use repeatedly in this paper.

**Lemma 4.6** (Key lemma). *Let  $C$  be a boundary component of  $S$  and  $\phi \in \text{Aut}(S, \partial S)$ . If there exists a (possibly immersed) geodesic arc  $\gamma \subset S$  that starts on  $C$  and satisfies  $T_C^m(\gamma) \geq \phi(\gamma) \geq T_C^M(\gamma)$  for some  $m, M \in \mathbb{Z}$  then  $m \leq c(\phi, C) \leq M$ .*

*Proof.* Assume contrary that  $M < c(\phi, C)$  then  $c(T_C^{-M}\phi, C) > 0$ . Propositions 3.1 and 3.2 of [20] imply that  $T_C^{-M}\phi$  is strictly right-veering with respect to  $C$ . Hence for any immersed geodesic arc  $\alpha$  which begins on  $C$  we have  $\alpha > T_C^{-M}\phi(\alpha)$ , hence  $T_C^M(\alpha) > \phi(\alpha)$ . This contradicts the assumption. The proof of  $m \leq c(\phi, C)$  is similar.  $\square$

Practically, in order to compute  $c(\phi, C)$  one may need to know the Nielsen-Thurston normal form for  $\phi$  and its invariant measured lamination. Proposition 4.5 and Lemma 4.6 provide effective methods to compute  $c(\phi, C)$  without using Nielsen-Thurston theory. Recall that in [27] Mosher proves that the mapping class group of  $S$  is automatic hence each element of  $\text{Aut}(S, \partial S)$  admits a normal form called *Mosher's normal form*.

**Theorem 4.7.** *Let  $S = S_{g,d}$  and  $D(S) = \max\{4g + 2, 4g + d - 3\}$ . Fix an integer  $N > D(S)(D(S) - 1)$ . Suppose that there exists a geodesic arc  $\gamma \subset S$  that starts on  $C$  and an integer  $M$  satisfying*

$$T_C^M(\gamma) \geq \phi^N(\gamma) > T_C^{M+1}(\gamma).$$

*Then the fractional Dehn twist coefficient has*

$$c(\phi, C) = \left[ \frac{M}{N}, \frac{M+1}{N} \right] \cap \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \{1, 2, \dots, D(S)\} \right\}.$$

*Moreover,  $c(\phi, C)$  can be computed in polynomial time with respect to the length of Mosher's normal form of  $\phi$ .*

*Proof.* By Lemma 4.6 and Proposition 4.4,  $M \leq c(\phi^N, C) = Nc(\phi, C) \leq M + 1$  so  $c(\phi, C) \in [\frac{M}{N}, \frac{M+1}{N}]$ . On the other hand, by Proposition 4.5,  $c(\phi, C) \in \{\frac{p}{q} \mid p \in \mathbb{Z}, q = 1, 2, \dots, D(S)\}$ . Since we choose  $N$  with  $N > D(S)(D(S) - 1)$ , the intersection

$$\left[ \frac{M}{N}, \frac{M+1}{N} \right] \cap \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \{1, 2, \dots, D(S)\} \right\}$$

consists of one rational number, which must be  $c(\phi, C)$ .

Next we show that  $c(\phi, C)$  is computable in polynomial time. We define a partial ordering  $<_\gamma$  on  $\text{Aut}(S, \partial S)$  by  $\phi \leq_\gamma \psi$  if  $\phi(\gamma) \geq \psi(\gamma)$ . As shown in [32, Theorem 2.1], this partial ordering is determined in linear time with respect to the length  $l(\phi)$  of Mosher's automatic normal form of  $\phi$  (see [27] for the definition) by using Mosher's automatic structure of  $\text{Aut}(S, \partial S)$ . By definition of Mosher's normal form, each generator  $x$  of Mosher's normal form satisfies  $c(\phi, C) = 0$  so we have an a priori estimate  $|c(\phi, C)| \leq$

$l(\phi)$ . This implies that the above integer  $M$  can be computable in polynomial time with respect to  $l(\phi)$ , hence so is  $c(\phi, C)$ .  $\square$

**Corollary 4.8.** *If there exists a (possibly immersed) geodesic arc  $\gamma \subset S$  starts on  $C \subset \partial S$  with  $T_C^m(\gamma) = \phi^N(\gamma)$  for some  $m, N \in \mathbb{Z}$  ( $N \neq 0$ ), then  $c(\phi, C) = \frac{m}{N}$ .*

**4.2. Alternative description of  $c(\phi, C)$ .** We give an alternative description of the fractional Dehn twist coefficient which appears to be natural from theoretical point of view. Let  $\pi : \tilde{S} \rightarrow S$  be the universal covering of  $S$ . Fix a base point  $* \in C \subset \partial S$  and its lift  $\tilde{*} \in \pi^{-1}(C) \subset \pi^{-1}(\partial S)$ . Let  $\tilde{C}$  be the connected component of  $\pi^{-1}(C)$  that contains  $\tilde{*}$ . Since  $S$  admits a hyperbolic metric there is an isometric embedding of  $\tilde{S}$  to the Poincaré disc  $\mathbb{H}^2$ . By attaching points at infinity to  $\tilde{S}$  we obtain a compact disc  $\overline{\tilde{S}} \subset \overline{\mathbb{H}^2}$ .

For a homeomorphism  $f : S \rightarrow S$  fixing the boundary pointwise we take the lift  $\tilde{f} : \tilde{S} \rightarrow \tilde{S}$  with  $\tilde{f}(\tilde{*}) = \tilde{*}$ . It uniquely extends to a homeomorphism  $\overline{\tilde{f}} : \overline{\tilde{S}} \rightarrow \overline{\tilde{S}}$ . The restriction  $\overline{\tilde{f}}|_{\partial \overline{\tilde{S}}}$  is an invariant of the mapping class  $[f] \in \mathcal{MCG}(S)$ . Since  $f = id$  on  $\partial S$  and  $\tilde{f}(\tilde{*}) = \tilde{*}$  the map  $\overline{\tilde{f}}$  fixes  $\tilde{C}$  pointwise. We identify  $\partial \overline{\tilde{S}} \setminus \tilde{C}$  with  $\mathbb{R}$  so that  $\overline{T_C}(x) = x + 1$  for all  $x \in \mathbb{R}$ . Let  $\widetilde{\text{Homeo}}^+(S^1)$  be the group of orientation-preserving homeomorphisms of  $\mathbb{R}$  that are lifts of orientation-preserving homeomorphisms of  $S^1$ . In other words,  $\widetilde{\text{Homeo}}^+(S^1)$  consists of elements of  $\text{Homeo}^+(\mathbb{R})$  that commute with the translation  $x \mapsto x + 1$ . Since  $f \circ T_C = T_C \circ f$  we can define a homomorphism  $\Theta_C : \mathcal{MCG}(S) \rightarrow \widetilde{\text{Homeo}}^+(S^1)$

by

$$\Theta_C([f]) = \overline{\tilde{f}}|_{\partial \overline{\tilde{S}} \setminus \tilde{C}}.$$

The map  $\Theta_C$  is called the *Nielsen-Thurston homomorphism* and is intensively studied in [34] to describe a total left-invariant ordering of  $\mathcal{MCG}(S)$ . It is known that  $\Theta_C$  is injective [34]. We note that  $\Theta_C$  depends on various choices such as hyperbolic metrics on  $S$  and identifications of  $\partial \overline{\tilde{S}} \setminus \tilde{C}$  with  $\mathbb{R}$ .

Let  $\tau : \widetilde{\text{Homeo}}^+(S^1) \rightarrow \mathbb{R}$  be the *translation number* defined by

$$\tau(h) = \lim_{N \rightarrow \infty} \frac{h^N(x) - x}{N} \quad (x \in \mathbb{R}).$$

It is well-known that the above limit exists and is independent of the choice of  $x \in \mathbb{R}$ . The fractional Dehn twist coefficient is related to the Nielsen-Thurston map as follows.

**Theorem 4.9.** (cf. [14, p.3]) *For  $\phi \in \text{Aut}(S, \partial S)$  we have  $c(\phi, C) = \tau(\Theta_C(\phi))$ .*

*Proof.* Let us take a geodesic  $\tilde{\gamma}$  in  $\tilde{S} \subset \mathbb{H}^2$  which joins  $\tilde{*}$  and  $x \in \partial \overline{\tilde{S}} \setminus \tilde{C} = \mathbb{R}$ . Denote  $\gamma = \pi(\tilde{\gamma})$ . For  $N > 0$  there exists an integer  $M(N)$  such that

$$T_C^{M(N)}(\gamma) \geq \phi^N(\gamma) \geq T_C^{M(N)+1}(\gamma).$$

This is equivalent to

$$\Theta_C(T_C^{M(N)})(x) \leq \Theta_C(\phi^N)(x) \leq \Theta_C(T_C^{M(N)+1})(x).$$

Recall that  $\Theta(T_C)$  translates  $x \mapsto x + 1$ , hence

$$x + M(N) \leq \Theta_C(\phi^N)(x) \leq x + M(N) + 1,$$

i.e.,

$$\frac{M(N)}{N} \leq \frac{\Theta_C(\phi^N)(x) - x}{N} \leq \frac{M(N) + 1}{N}.$$

By Theorem 4.7 as  $N \rightarrow \infty$  both  $\frac{M(N)}{N}$  and  $\frac{M(N)+1}{N}$  converge to  $c(\phi, C)$  and the middle term converges to  $\tau(\Theta_C(\phi))$ , so we obtain  $c(\phi, C) = \tau(\Theta_C(\phi))$ .  $\square$

Since the translation number  $\tau : \widetilde{\text{Homeo}}^+(S^1) \rightarrow \mathbb{R}$  is a homogeneous quasi-morphism of defect 1, we get the following.

**Corollary 4.10.** *The fractional Dehn twist coefficient w.r.t.  $C$  defines a homogeneous quasi-morphism*

$$c(\cdot, C) : \text{Aut}(S, \partial S) \rightarrow \mathbb{Q}(\mathbb{R})$$

of defect 1. That is,

$$|c(\phi\psi, C)c(\phi, C) - c(\psi, C)| \leq 1$$

and

$$c(\phi^N, C) = Nc(\phi, C)$$

hold for all  $\phi, \psi \in \mathcal{MCG}(S)$  and  $N \in \mathbb{Z}$ .

## 5. ESTIMATES OF FRACTIONAL DEHN TWIST COEFFICIENT FROM OPEN BOOK FOLIATION

In this section we give estimates of the fractional Dehn twist coefficient from the open book foliation. The notion of strong essentiality plays an important role. Lemma 5.1, a special case of Theorem 5.3, gives a simple but still useful estimate. The proof shows a basic idea how the fractional Dehn twist coefficient and the open book foliation are related to each other.

**Lemma 5.1.** *Let  $v$  be an elliptic point of  $\mathcal{F}_{ob}(F)$  lying on a binding component  $C \subset \partial S$ . Assume that  $v$  is strongly essential and there are no a-arcs around  $v$ . Let  $p$  (resp.  $n$ ) be the number of positive (resp. negative) hyperbolic points that are joined with  $v$  by a singular leaf.*

(1) *If  $\text{sgn}(v) = +1$  then*

$$-n \leq c(\phi, C) \leq p.$$

(2) *If  $\text{sgn}(v) = -1$  then*

$$-p \leq c(\phi, C) \leq n.$$

*Proof.* We prove the case  $\text{sgn}(v) = -1$ . We remark that since  $\text{sgn}(v) = -1$  any regular leaf that ends at  $v$  is a b-arc, so the assumption that around  $v$  there are no a-arcs is automatically satisfied. Similar arguments hold for the positive case.

Let  $h_1, \dots, h_{n+p}$  be the hyperbolic points around  $v$ , and  $S_{t_i}$  be the singular fiber that contains  $h_i$ . With no loss of generality we may assume  $0 < t_1 < \dots < t_{n+p} < 1$ . For  $t \neq t_1, \dots, t_{n+p}$  let  $b_t$  denote the b-arc in  $S_t$  that ends at  $v$ . Since  $v$  is strongly essential, all the  $b_t$  are strongly essential.

Suppose that  $\text{sgn}(h_1) = +1$ . Let  $\gamma \subset S_0 \setminus (S_0 \cap F)$  be the describing arc for  $h_1$ . At least one of the endpoints of  $\gamma$  lies on  $b_0$ , which we call  $v'$  (if both the endpoints lie on  $b_0$ , pick the one closer to  $v$ ). We isotope  $\gamma$  in  $S_0 \setminus (S_0 \cap F)$  by sliding  $v'$  along  $b_0$  until it reaches  $v$ . Since  $\text{sgn}(v) = -1$  and  $\text{sgn}(h_1) = +1$ , near  $v$  the arc  $\gamma$  lies strictly on the

right of  $b_0$  in  $S_0$ . Hence  $b_0 > \gamma$ . See Figure 8. On the other hand, since the interiors

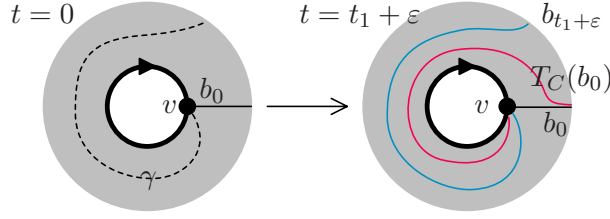


FIGURE 8. Passing a hyperbolic point twists a b-arc at most once.

of  $\gamma$  and  $b_0$  are disjoint and  $b_0$  is strongly essential (i.e.,  $b_0 \neq T_C(b_0)$ ),  $\gamma$  lies strictly on the left side of  $T_C(b_0)$ . We have  $\gamma > T_C(b_0)$ . After passing the critical time  $t_1$  we may identify  $b_{t_1+\epsilon}$  with  $\gamma$  for sufficiently small  $\epsilon > 0$ . Hence we get

$$b_0 > b_{t_1+\epsilon} > T_C(b_0).$$

Similarly, if  $\text{sgn}(h_1) = -1$  we obtain

$$T_C^{-1}(b_0) > b_{t_1+\epsilon} > b_0.$$

Fix  $1 \leq k \leq p+n$ . Suppose that  $\alpha$  of the hyperbolic points  $h_1, \dots, h_k$  are negative and  $\beta = k - \alpha$  of them are positive. If  $T_C^{-\alpha}(b_0) > b_{t_k+\epsilon} > T_C^{\beta}(b_0)$  then the above argument implies that  $T_C^{-\alpha}(b_0) > b_{t_k+\epsilon} = b_{t_{k+1}-\epsilon} > b_{t_{k+1}+\epsilon} > T_C(b_{t_{k+1}-\epsilon}) = T_C(b_{t_k+\epsilon}) > T_C^{\beta+1}(b_0)$  when  $\text{sgn}(h_{k+1}) = +1$ . Hence

$$T_C^{-\alpha}(b_0) > b_{t_{k+1}+\epsilon} > T_C^{\beta+1}(b_0) \quad \text{if } \text{sgn}(h_{k+1}) = +1,$$

$$T_C^{-(\alpha+1)}(b_0) > b_{t_{k+1}+\epsilon} > T_C^{\beta}(b_0) \quad \text{if } \text{sgn}(h_{k+1}) = -1.$$

By induction on  $k$  we conclude

$$T_C^{-n}(b_0) > b_1 = \phi^{-1}(b_0) > T_C^p(b_0).$$

Since the elliptic point  $v$  is strongly essential, the b-arc  $b_0$  is strongly essential, i.e.,  $b_0$  is an essential arc in  $S$ . Lemma 4.6 implies  $-n \leq c(\phi^{-1}, C) \leq p$ . With Proposition 4.4-(1) we obtain the desired estimate.  $\square$

**Remark.** In [22], [23] we used the similar arguments to relate the valence of a vertex in the braid foliation (which corresponds to the number of hyperbolic singular points around a elliptic singular points in open book foliation) and the Dehornoy floor, an integer-valued complexity of braids defined by the Dehornoy ordering of the braid groups which roughly corresponds to the absolute value of the fractional Dehn twist coefficient. Lemma 5.1 can be seen as a generalization of the argument in [22] and [23].

Below is an immediate consequence of Lemma 5.1.

**Theorem 5.2.** *Let  $v_1, \dots, v_n$  be the strongly essential elliptic points on a binding component  $C$  of the open book  $(S, \phi)$  such that all the regular leaves ending on  $v_i$  are b-arcs.*



Let  $p_i$  (resp.  $n_i$ ) be the number of positive (resp. negative) hyperbolic points connected to  $v_i$  by a singular leaf. Define the upper bound function

$$U(v_i) = \begin{cases} p_i & \text{if } \mathbf{sgn}(v_i) = +1, \\ n_i & \text{if } \mathbf{sgn}(v_i) = -1, \end{cases}$$

and the lower bound function

$$L(v_i) = \begin{cases} -n_i & \text{if } \mathbf{sgn}(v_i) = +1, \\ -p_i & \text{if } \mathbf{sgn}(v_i) = -1. \end{cases}$$

Then the fractional Dehn twist coefficient has

$$(5.1) \quad \max_{i=1, \dots, n} L(v_i) \leq c(\phi, C) \leq \min_{i=1, \dots, n} U(v_i).$$

Lemma 5.1 and Theorem 5.2 above require only local information of the open book foliation, namely, the number of hyperbolic points around a *single* strongly essential elliptic point. In Theorem 5.3 below we examine *more than one* strongly essential elliptic points at the same time, and get a sharper estimate of  $c(\phi, C)$ .

Let  $\lceil x \rceil \in \mathbb{Z}$  be the *ceiling* of  $x \in \mathbb{R}$ , that is, the smallest integer greater than or equal to  $x$ . For a function  $f : \mathbb{N} \rightarrow \mathbb{R}$ , we define

$$F(f) = \inf_{m \in \mathbb{N}} \left\{ \frac{\lceil f(m) \rceil}{m} \right\} \in \mathbb{R}.$$

**Theorem 5.3.** *Let  $v_1, \dots, v_n \in \mathcal{F}_{ob}(F)$  be strongly essential elliptic points. Assume that all of them lie on the same component  $C$  of  $\partial S$  and that all the regular leaves ending on  $v_i$  are  $b$ -arcs. Let  $N$  (resp.  $P$ ) be the total number of negative (resp. positive) hyperbolic points that are connected to at least one of  $v_1, \dots, v_n$  by a singular leaf. Let  $f_{\pm} : \mathbb{N} \rightarrow \mathbb{Q}$  be a map defined by*

$$f_{-}(m) = \begin{cases} \frac{Nm}{n} - \frac{(n-1)^2}{4n^2} & (n : \text{odd}) \\ \frac{Nm}{n} - \frac{n-2}{4n} & (n : \text{even}) \end{cases}$$

and

$$f_{+}(m) = \begin{cases} \frac{Pm}{n} - \frac{(n-1)^2}{4n^2} & (n : \text{odd}) \\ \frac{Pm}{n} - \frac{n-2}{4n} & (n : \text{even}). \end{cases}$$

(1) *If  $\mathbf{sgn}(v_1) = \mathbf{sgn}(v_2) = \dots = \mathbf{sgn}(v_n) = -1$ , then*

$$(5.2) \quad -F(f_{+}) \leq c(\phi, C) \leq F(f_{-}).$$

(2) *If  $\mathbf{sgn}(v_1) = \mathbf{sgn}(v_2) = \dots = \mathbf{sgn}(v_n) = +1$  and all the regular leaves from  $v_i$  are  $b$ -arcs, then*

$$(5.3) \quad -F(f_{-}) \leq c(\phi, C) \leq F(f_{+}).$$

**Remark.** Lemma 5.1 is a corollary of Theorem 5.3 for the case  $n = 1$ . If  $n \geq 2$  it depends which estimate of (5.1), (5.2) or (5.3) is the sharpest.

*Proof.* We show the upper bound of  $c(\phi, C)$  for Case (1). The rest of the bounds can be obtained similarly.

Let  $A = F \cap S_0$  be the multi-curve on  $S = S_0$ . We cut  $F \subset M$  along  $S_0$  to get a properly embedded oriented surface  $\Sigma$  in  $\overline{M} \setminus \overline{S_0} \simeq S \times [0, 1] / \sim_\partial$  where “ $\sim_\partial$ ” is an equivalence relation  $(x, t) \sim_\partial (x, 0)$  for  $x \in \partial S$  and  $t \in [0, 1]$ . We orient  $A$  so that

$$\Sigma \cap S_0 = -A, \quad \Sigma \cap S_1 = \phi^{-1}(A).$$

Fix an integer  $m \geq 1$ . For  $i = 0, \dots, m-1$ , let

$$\Phi_i : S \times [0, 1] / \sim_\partial \rightarrow S \times [i, i+1] / \sim_\partial$$

be a map defined by

$$\Phi_i(x, t) = (\phi^{-i}(x), t + i).$$

Let

$$\Sigma_m = \Sigma \cup \Phi_1(\Sigma) \cup \dots \cup \Phi_{m-1}(\Sigma) \subset S \times [0, m] / \sim_\partial,$$

a properly embedded surface. Consider a natural quotient map

$$\pi_m : S \times [0, m] / \sim_\partial \rightarrow M_{(S, \phi^m)}$$

identifying  $(x, m)$  with  $(\phi^m(x), 0)$  for  $x \in \text{Int}(S)$ . Note that

$$\Sigma_m \cap S_0 = -A \quad \text{and} \quad \Sigma_m \cap S_m = \phi^{-m}(A).$$

Hence we obtain a surface  $F_m := \pi_m(\Sigma_m) \subset M_{(S, \phi^m)}$ .

The manifold  $M_{(S, \phi^m)}$  is the cyclic  $m$ -fold branched cover of  $M_{(S, \phi)}$  with branch locus the binding of the open book. Likewise the surface  $F_m$  is the cyclic  $m$ -fold branched cover of  $F$ . By abuse of notation let  $v_1, \dots, v_n \in F_m$  denote the lifts of the branched points  $v_1, \dots, v_n \in F$ . By construction, the lifts  $v_1, \dots, v_n$  are also strongly essential negative elliptic points in  $\mathcal{F}_{ob}(F_m)$  and connected to  $Nm$  negative hyperbolic points,  $h_1, \dots, h_{Nm}$ , by a singular leaf. We assume that  $h_k$  lies on the singular fiber  $S_{t_k}$  with

$$0 < t_1 < t_2 < \dots < t_N < 1 \quad \text{and} \quad t_{lN+j} = l + t_j.$$

For  $i = 1, \dots, n$  and  $t \neq t_k$ , we denote the b-arc in  $S_t$  that ends at  $v_i$  by  $b_t^i$ . As in the proof of Lemma 5.1, we compute the upper bound of  $c(\phi^m, C)$  by comparing  $b_0^i$  and  $b_m^i = \phi^{-m}(b_0^i)$ . To this end we introduce *twisting* of  $b_t^i$  and *total twisting* on  $S_t$ :

Consider a natural projection  $\mathcal{P} : M_{(S, \phi^m)} \rightarrow S$  defined by  $\mathcal{P}(x, t) = x$ . (Below, for sake of simplicity we freely identify an arc in a page  $S_t$  and its image under  $\mathcal{P}$ .) If  $b_t^i > b_0^i$  near  $v_0$ , there exists an integer  $l_{i,j} = l_{i,j}(t) \in \mathbb{Z}_{\leq 0}$  such that the geometric intersection number

$$i(b_t^i, T_C^l(b_0^j)) := i(\mathcal{P}(b_t^i), \mathcal{P}(T_C^l(b_0^j)))$$

is minimized when  $l = l_{i,j}$ . We define  $tw(b_t^i)$ , the *twisting* of  $b_t^i$ , by

$$tw(b_t^i) := \begin{cases} 0 & \text{if } b_0^i \geq b_t^i \text{ near } v_i, \\ 1 + \sum_{j=1}^n \left( i(b_t^i, b_0^j) - i(b_t^i, T_C^{l_{i,j}}(b_0^j)) \right) & \text{if } b_t^i > b_0^i \text{ near } v_i. \end{cases}$$

By definition,  $tw(b_t^i) \geq 0$ . For each  $i$ ,  $tw(b_t^i)$  is a step function of  $t$  and the value possibly increases at  $t = t_1, \dots, t_{Nm}$  because  $\text{sgn}(h_1) = \dots = \text{sgn}(h_{Nm}) = -1$  and  $\text{sgn}(v_i) = -1$ . Morally,  $tw(b_t^i)$  measures how much the diffeomorphism  $\phi^{-m}$  turns  $b_0^i$  to the *left* near  $v_i$  on the page  $S_t$ .

Here is an alternative explanation. Assume that  $b_t^i$  and the multi-curve  $b_0^1 \cup \dots \cup b_0^n$  attain the minimal geometric intersection number. We consider the subset

$$\mathcal{I}_t^i \subset \text{int}(b_t^i) \cap (b_0^1 \cup \dots \cup b_0^n)$$

that represents the  $T_C^{-1}$ -factor contributing to  $\phi^{-m}$ . We have

$$tw(b_t^i) = |\mathcal{I}_t^i| + 1,$$

where “+1” corresponds to the end point  $v_i$ . Since  $\mathcal{I}_t^i$  locates near  $C$  we can take an annular collar neighborhood  $N(C)$  of  $C$  containing  $\mathcal{I}_t^1 \cup \dots \cup \mathcal{I}_t^n$  such that the term

$$\sum_{j=1}^n \left( i(b_t^i, b_0^j) - i(b_t^i, T_C^{l_{i,j}}(b_0^j)) \right) = |\mathcal{I}_t^i| = |\text{int}(b_t^i) \cap (b_0^1 \cup \dots \cup b_0^n) \cap N(C)|$$

counts the number of intersection points lying in  $N(C)$ . See Figure 9, where the gray

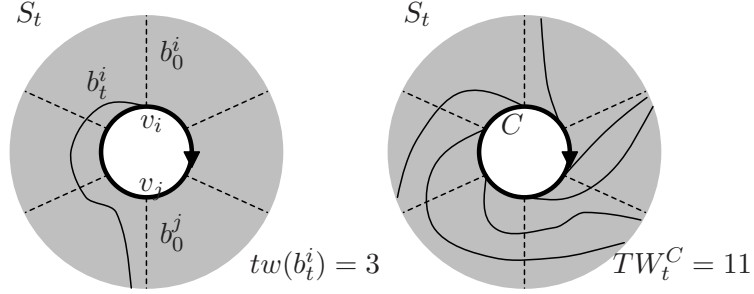


FIGURE 9. The twisting  $tw(b_t^i)$  and the total twisting  $TW_t^C$ .

area represents  $N(C)$  and the dashed lines denote the multi-curve  $(b_0^1 \cup \dots \cup b_0^n) \cap N(C)$ . When  $t = m$ , from the definition of  $tw(b_m^i)$  for every  $i = 1, \dots, n$  we have:

$$T_C^{-\nu}(b_0^i) \geq b_m^i \quad \text{where } \nu = \left\lceil \frac{tw(b_m^i)}{n} \right\rceil \geq 0.$$

i.e.,

$$b_0^i = \phi^m(b_m^i) \geq T_C^\nu(b_m^i).$$

Hence Lemma 4.6 implies that

$$(5.4) \quad c(\phi^m, C) \leq \left\lceil \frac{tw(b_m^i)}{n} \right\rceil \quad \text{for all } i = 1, \dots, n.$$

The *total twisting*  $TW_t^C$  on the fiber  $S_t$  along the boundary  $C$  is defined by:

$$(5.5) \quad TW_t^C := \sum_{i=1}^n tw(b_t^i)$$

See the right picture in Figure 9. As we have seen in the proof of Lemma 5.1 positive hyperbolic points do not contribute to the total twisting, so in the next two observations we concentrate on the effect of negative hyperbolic points  $h_1, \dots, h_{mN}$ :

**Observation 5.4.** For each  $j = 1, \dots, mN$  the hyperbolic point  $h_j$  increases the total twisting by at most  $n$ , namely:

$$0 \leq TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C \leq n.$$

*Proof.* Since  $\text{sgn}(h_j) = -1$  and  $\text{sgn}(v_i) = -1$ , we have  $0 \leq TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C$ .

Let  $\gamma_j$  denote the describing arc for  $h_j$ . We may assume that at least one of the endpoints of  $\gamma_j$  stays in the region  $N(C)$ . We have

$$TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C = |\gamma_j \cap (b_0^1 \cup \dots \cup b_0^n) \cap N(C)| \leq n.$$

□

In general, the equality  $TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C = n$  may hold when the b-arcs  $b_{t_j-\varepsilon}^1, \dots, b_{t_j-\varepsilon}^n$  get out of  $N(C)$  through (1) a single region between consecutive  $b_0^{r_\circ}$  and  $b_0^{r_\circ+1}$ , or (2) two regions as depicted in Figure 10.

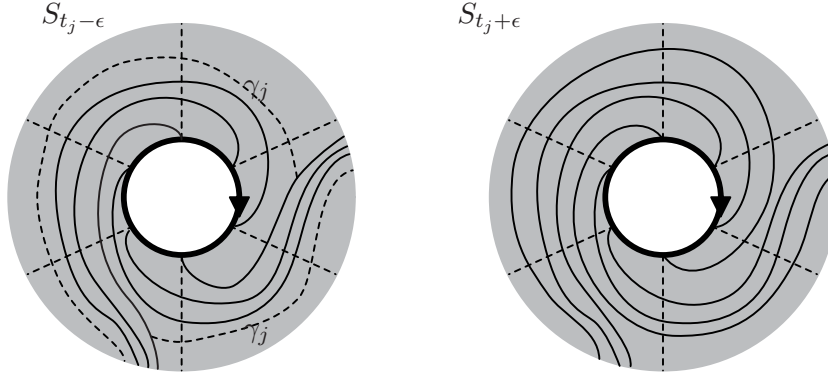


FIGURE 10. (Observation 5.4)

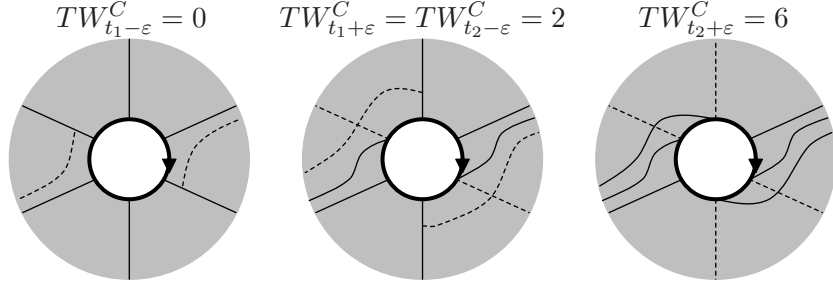
**Observation 5.5.** For each  $j = 1, \dots, \lfloor \frac{n}{2} \rfloor$  the  $j$ -th negative hyperbolic point  $h_j$  increases the total twisting by at most  $2j$ , that is,

$$0 \leq TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C \leq 2j.$$

*Proof.* Suppose that the describing arc  $\gamma_1$  for  $h_1$  joins  $b^{i_\circ}$  and  $b^{i_\bullet}$  for some  $i_\circ, i_\bullet \in \{1, \dots, n\}$ .  $\gamma_j$  joins  $b^{i_\circ+j-1}$  and  $b^{i_\bullet+j-1}$  for all  $j = 1, \dots, \lfloor \frac{n}{2} \rfloor$  if and only if the equality  $TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C = 2j$  holds. See Figure 11, where the cases  $j = 1, 2$  are depicted. □

From Observations 5.4 and 5.5, letting  $k = \lfloor \frac{n}{2} \rfloor$  we have:

$$\begin{aligned} TW_m^C &\leq 2 + 4 + \dots + 2k + (Nm - k)n \\ &= \begin{cases} Nm - \frac{(n-1)^2}{4} & n: \text{ odd} \\ Nm - \frac{n(n-2)}{4} & n: \text{ even} \end{cases} \end{aligned}$$

FIGURE 11. (Observation 5.5) when  $j = 1$  and  $2$ .

Thus by (5.5) we have:

$$\min_{i=1,\dots,n} \{tw(b_m^i)\} \leq \begin{cases} Nm - \frac{(n-1)^2}{4n} & n: \text{ odd} \\ Nm - \frac{(n-2)}{4} & n: \text{ even} \end{cases}$$

By (5.4) we have for all  $m \in \mathbb{N}$ :

$$m \cdot c(\phi, C) = c(\phi^m, C) \leq \begin{cases} \left\lceil \frac{Nm}{n} - \frac{(1-n)^2}{4n^2} \right\rceil & n: \text{ odd} \\ \left\lceil \frac{Nm}{n} - \frac{n-2}{4n} \right\rceil & n: \text{ even} \end{cases}$$

□

## 6. NON-RIGHT-VEERINGNESS AND OPEN BOOK FOLIATION

The notion of right-veeringness is closely related to tight/overtwisted-ness of the contact structure supported by the open book: Recall a theorem of Honda-Kazez-Matić [20, Theorem 1.1], (cf. [24, Theorem 6.7] for an alternative proof based on open book foliation).

**Theorem 6.1.** *If  $\phi \in \text{Aut}(S, \partial S)$  is not right-veering then  $(S, \phi)$  supports an overtwisted contact structure.*

In this section we characterize non-right-veering monodromy via open book foliation, especially by a special transverse overtwisted disc. The results highlight the fact that right-veeringness in general does not imply tightness of the compatible contact structure.

**Theorem 6.2.** *A diffeomorphism  $\phi \in \text{Aut}(S, \partial S)$  is not right-veering if and only if there exists a transverse overtwisted disc  $D$  in  $(S, \phi)$  whose open book foliation has exactly one negative elliptic point. See Figure 12.*

*Proof.* ( $\Rightarrow$ ) If  $\phi$  is non-right-veering then there exists an essential properly embedded arc  $\gamma$  in  $S$  such that  $\phi(\gamma) > \gamma$ . As shown in the proof of [24, Theorem 6.7] we can construct a transverse overtwisted disc with exactly one negative elliptic point.

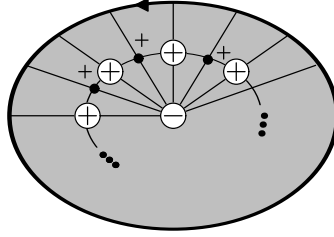


FIGURE 12. A transverse overtwisted disc with one negative elliptic point.

( $\Leftarrow$ ) Let  $v$  be the negative elliptic point in  $\mathcal{F}_{ob}(D)$ , and  $0 < t_1 < \dots < t_k < 1$  be the numbers such that the page  $S_{t_i}$  contains a positive hyperbolic point  $h_i$  of  $\mathcal{F}_{ob}(D)$ . Let  $b_t \subset S_t$  ( $t \neq t_i$ ) be the b-arc emanating from  $v$ .

**Claim 6.3.**  $v$  is strongly essential.

*Proof.* Assume contrary that  $v$  is not strongly essential. We may suppose that the b-arc  $b_0$  is boundary parallel in  $S_0$ . Let  $w$  be the positive elliptic point that is the other end point of  $b_0$  so that the arc  $\overline{vw} \subset C$  cobounds a disc  $\Delta_0 \subset S_0$  with  $b_0$ . We may choose  $b_0$  so that the disc  $\Delta_0$  is *minimal* in the sense that if any b-arc  $b_t$  cobounds a disc  $\Delta_t$  with  $C$  then  $\Delta_t \not\subset \Delta_0$  under the natural projection  $\mathcal{P} : S_t \rightarrow S$ . In the following for simplicity of notation we denote the image of arcs and discs under  $\mathcal{P}$  by the same symbol. There are two cases to consider:

- (i): The disc  $\Delta_0$  lies on the right side of  $b_0$  as we walk from  $v$  to  $w$ .
- (ii): The disc  $\Delta_0$  lies on the left side of  $b_0$  as we walk from  $v$  to  $w$ .

Let us consider the case (i). Since  $\text{sgn}(h_1) = +1$  and  $\text{sgn}(v) = -1$ , the describing arc for  $h_1$  (hence  $b_{t_1+\varepsilon}$ ) lies on the interior of  $\Delta_0$ . That is, the disc  $\Delta_{t_1+\varepsilon}$  co-bounded by  $b_{t_1+\varepsilon}$  sits inside  $\Delta_0$ , which contradicts the minimality of  $\Delta_0$ .

A similar argument works for case (ii).  $\square$

Since all the hyperbolic points have  $\text{sgn} = +1$ , using the same argument as in the proof of Lemma 5.1 we get

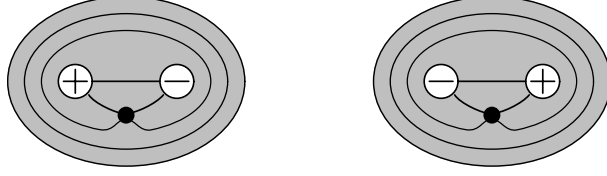
$$\phi(b_1) = b_0 > b_{t_1+\varepsilon} > b_{t_2+\varepsilon} > \dots > b_{t_k+\varepsilon} = b_1 \text{ near } v.$$

Claim 6.3 guarantees that  $b_1$  is an essential arc in  $S_1$ . Thus  $\phi$  is not right-veering with respect to  $C$ . This concludes Theorem 6.2.  $\square$

The open book foliation provides more sufficient conditions for non-right-veering-ness. Recall that a bc-annulus  $R$  is *degenerated* if the two b-arc boundaries are identified.

**Theorem 6.4.** *If there exists a (possibly closed) surface  $F$  in an open book  $(S, \phi)$  whose open book foliation contains a degenerated bc-annulus  $R$  with essential c-circles (cf. Figure 13) then  $\phi$  is not right-veering, hence  $(S, \phi)$  supports an overtwisted contact structure.*

*Proof.* We may assume that  $R$  is as in the right sketch of Figure 13. Let  $S_{t_0}$  be the fiber on which the unique hyperbolic point  $h \in \mathcal{F}_{ob}(R)$  lies. Let  $v^+$  and  $v^-$  be the positive and the negative elliptic singular point of  $\mathcal{F}_{ob}(R)$ .

FIGURE 13. Degenerated  $bc$ -annulus  $R$ .

**Claim 6.5.**  $v^\pm$  are strongly essential.

*Proof.* Assume contrary that  $v^\pm$  are not strongly essential, i.e., for any  $t \neq t_0$  the b-arc  $b_t \subset R \cap S_t$  cobounds a disc  $\Delta_t \subset S_t$ . For  $t \in (t_0 - 2\varepsilon, t_0)$  let  $c_t \subset R \cap S_t$  denote the c-circle. There are two cases to consider:

- (1) If  $c_{t_0-\varepsilon} \subset \Delta_{t_0-\varepsilon}$  then  $c_{t_0-\varepsilon}$  bounds a disc  $X \subset \Delta_{t_0-\varepsilon} \subset S_{t_0-\varepsilon}$ .
- (2) If  $c_{t_0-\varepsilon} \subset (S_{t_0-\varepsilon} \setminus \Delta_{t_0-\varepsilon})$  then  $c_{t_0-\varepsilon} \# b_{t_0-\varepsilon} \simeq b_{t_0+\varepsilon}$ , so  $c_{t_0-\varepsilon}$  bounds a disc  $X \subset S_{t_0-\varepsilon} \setminus \Delta_{t_0-\varepsilon}$ .

Suppose that  $m = |X \cap \partial R|$  and  $n = |\Delta_{t_0-\varepsilon} \cap \partial R|$ . Because of the hyperbolic point  $h$  we have

$$|\Delta_{t_0+\varepsilon} \cap \partial R| = \begin{cases} n - m & \text{for case (1),} \\ n + m & \text{for case (2).} \end{cases}$$

Since  $\Delta_t$  has no interaction with rest of the world when  $t \neq t_0$ , we have

$$|\Delta_{t_0-\varepsilon} \cap \partial R| = |\Delta_{t_0+\varepsilon} \cap \partial R|.$$

Hence  $m = 0$ , i.e.,  $c_t$  is inessential, which contradicts our assumption.  $\square$

Since the c-circles  $c_t$  are essential, if  $\text{sgn}(R) = \pm 1$  we have

$$\phi(b_1) = b_0 = b_{t_0-\varepsilon} > b_{t_0+\varepsilon} = b_1 \quad \text{near } v^\mp.$$

Let  $C^\pm \subset \partial S$  be the binding component of the open book that contains  $v^\pm$  respectively. By Claim 6.5 the b-arc  $b_1$  is strongly essential, hence  $b_1$  is an essential arc in  $S$ . Applying Lemma 4.6-(1) we obtain  $c(\phi, C^\mp) < 0$  when  $\text{sgn}(R) = \pm 1$ , respectively, i.e.,  $\phi$  is not right veering.  $\square$

**Corollary 6.6.** Let  $D \subset (S, \phi)$  be a disc with an open book foliation  $\mathcal{F}_{ob}(D)$  satisfying

- (1)  $sl(\partial D, [D]) = 1$ ,
- (2)  $\mathcal{F}_{ob}(D)$  contains c-circles, and
- (3)  $e_-(\mathcal{F}_{ob}(D)) = 1$ .

Then  $\phi \in \text{Aut}(S, \partial S)$  is non-right-veering.

*Proof.* Since  $D \simeq D^2$ ,  $\mathcal{F}_{ob}(D)$  contains c-circles and  $e_-(\mathcal{F}_{ob}(D)) = 1$ , the region decomposition of  $D$  contains one degenerated bc-annulus,  $R$ , and  $D \setminus R$  contains one (possibly degenerated) ac-annulus and aa-tiles.

**Claim 6.7.** The c-circles in  $\mathcal{F}_{ob}(D)$  are essential.

*Proof.* We assume that  $\mathcal{F}_{ob}(R)$  is as shown in the right sketch of Figure 13 (the same arguments apply to the left sketch case). A possibly degenerated ac-annulus surrounds  $R$  as shown in Figure 14. We may assume that the hyperbolic point of the ac-annulus



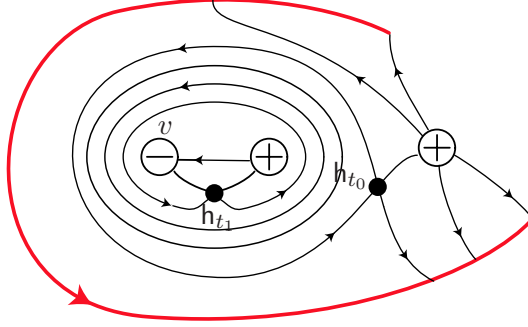


FIGURE 14. An ac-annulus surrounds a degenerated bc-annulus.

(resp.  $R$ ) lives on the page  $S_{t_0}$  (resp.  $S_{t_1}$ ) for some  $0 < t_0 < t_1 < 1$  and we name it  $h_{t_0}$  (resp.  $h_{t_1}$ ). Proposition 2.5 and the assumptions (1) and (3) imply that  $e_+(\mathcal{F}_{ob}(D)) = h_+(\mathcal{F}_{ob}(D))$  and  $h_-(\mathcal{F}_{ob}(D)) = 0$ . Therefore  $\text{sgn}(h_{t_0}) = \text{sgn}(h_{t_1}) = +1$ .

Suppose contrary that c-circles  $c_t \subset (D \cap S_t)$  for  $t_0 < t < t_1$  are inessential. For a very small  $\varepsilon > 0$  since  $\text{sgn}(h_{t_0}) = +1$  the circle  $c_{t_0+\varepsilon}$  bounds a disc in  $S_{t_0+\varepsilon}$  on its left side with respect to the orientation of  $c_{t_0+\varepsilon}$ . On the other hand, since  $\text{sgn}(h_{t_1}) = +1$  the circle  $c_{t_1-\varepsilon}$  bounds a disc in  $S_{t_1-\varepsilon}$  on the right side of  $c_{t_1-\varepsilon}$ . Since  $\{c_t \mid t_0 < t < t_1\}$  is a continuous family of oriented circles they can bound discs only on the same side, hence we get a contradiction.  $\square$

Claim 6.7 and Theorem 6.4 imply that  $\phi$  is not right-veering.  $\square$

## 7. TOPOLOGY OF OPEN BOOK MANIFOLDS

In this section we apply technique of the open book foliation to obtain results in topology of 3-manifolds. Throughout the section let  $(S, \phi)$  be an open book decomposition of a 3-manifold  $M \cong M_{(S, \phi)}$ . The main results are the following two:

**Theorem 7.1.** *Assume that;*

- (1)  $|c(\phi, C)| > 3$  for every boundary component  $C$  of  $S$ , or
- (2)  $\partial S$  is connected, and  $|c(\phi, \partial S)| > 1$ .

*Then the 3-manifold  $M$  is irreducible.*

**Theorem 7.2.** *Assume that  $\phi \in \text{Aut}(S, \partial S)$  is of irreducible type and that;*

- (1)  $|c(\phi, C)| > 4$  for every boundary component  $C$  of  $S$ , or
- (2)  $\partial S$  is connected, and  $|c(\phi, \partial S)| > 1$ .

*Then the 3-manifold  $M$  is irreducible and atoroidal.*

In order to prove the above theorems, we establish estimates of the fractional Dehn twist coefficient from topological data of incompressible closed surfaces in  $M$ .

**Theorem 7.3** (General case). *Suppose that there exists a closed, oriented incompressible genus  $g$  surface  $F$  in  $M$  which admits an essential open book foliation  $\mathcal{F}_{ob}(F)$ .*

- (1) *If  $g = 0$  and  $F$  intersects the binding, then there exists a boundary component  $C$  such that  $|c(\phi, C)| \leq 3$ .*

- (2) If  $g > 0$  and  $F$  intersects the binding in  $2n$  ( $n > 0$ ) points, then there exists a boundary component  $C$  such that  $c(\phi, C) \leq 4 + \lfloor \frac{4g-4}{n} \rfloor$ .

If  $S$  has connected boundary we get even sharper estimates:

**Theorem 7.4** (Connected binding case). *Under the same setting of Theorem 7.3, assume further that the binding of  $(S, \phi)$  is connected and  $F$  intersects the binding  $B = \partial S$  in  $2n$  ( $n > 0$ ) points. Then*

$$|c(\phi, \partial S)| \leq \inf_{m \in \mathbb{N}} g(m)$$

where  $g : \mathbb{N} \rightarrow \mathbb{Q}$  is a map defined by:

$$g(m) = \begin{cases} \frac{1}{m} \left\lceil \frac{(g-1+n)m}{n} - \frac{(n-1)^2}{4n^2} \right\rceil & (n: \text{ odd}) \\ \frac{1}{m} \left\lceil \frac{(g-1+n)m}{n} - \frac{n-2}{4n} \right\rceil & (n: \text{ even}) \end{cases}$$

In particular, if  $g = 0$  we have

$$-1 \leq c(\phi, \partial S) \leq 1,$$

and otherwise

$$-g \leq c(\phi, \partial S) \leq g.$$

Once we have Theorems 7.3 and 7.4 in hand, we can prove the main results easily:

*Proof of Theorem 7.1.* Assume that  $M$  is reducible. There exists a sphere  $\mathcal{S}$  which does not bound a 3-ball in  $M$ . In particular,  $\mathcal{S}$  is incompressible. By Theorem 3.2 with isotopy  $\mathcal{S}$  can admit an essential open book foliation. By the Euler characteristic argument,  $(e_+ + e_-) - (h_+ + h_-) = \chi(\mathcal{S}) = 2$ , we know that  $\mathcal{F}_{ob}(\mathcal{S})$  has elliptic points, i.e.,  $\mathcal{S}$  intersects the binding. Now Theorem 7.3-(1) together with Theorem 7.4 ( $g = 0$  case) yields the contrapositive of the statement of Theorem 7.1.  $\square$

*Proof of Theorem 7.2.* By Theorem 7.1 we know that  $M$  is irreducible so it remains to show that  $M$  is atoroidal. Assume contrary that  $M$  contains an incompressible torus  $T$ . We isotope  $T$  so that it admits an essential open book foliation. Theorems 7.3-(2) and 7.4 ( $g > 0$  case) guarantee that  $T$  does not intersect the binding, i.e.,  $\mathcal{F}_{ob}(T)$  contains no elliptic points. Since  $(e_+ + e_-) - (h_+ + h_-) = \chi(T) = 0$ ,  $\mathcal{F}_{ob}(T)$  contains no hyperbolic points as well and all the leaves are c-circles.

Hence the sets of c-circles  $T \cap S_1$  and  $T \cap S_0$  are isotopic in  $S$ . But the definition of open book decomposition imposes  $\phi(T \cap S_1) = T \cap S_0$ . Therefore  $\phi(T \cap S_1) = T \cap S_1$  which contradicts the assumption that  $\phi$  is irreducible.  $\square$

Before proving Theorems 7.3 and 7.4, we list some observations:

**Observation 7.5.** When  $F$  is an incompressible closed surface in  $M$ , there are various properties which make arguments simple.

- (1) Since  $F$  has no boundary, every essential b-arc is automatically strongly essential, hence there is no distinction between essentiality and strong essentiality for b-arcs. In particular, any b-arc in an essential open book foliation is strongly essential.

- (2)  $\mathcal{F}_{ob}(F)$  does not have a-arcs. Thus, the region decomposition of  $\mathcal{F}_{ob}(F)$  consists only of three types; bb-tile, bc-annulus, and cc-pants.
- (3) Since the binding  $B = \partial S$  is null-homologous in  $M$ , the algebraic intersection of  $F$  with  $B$  is zero. Hence the numbers of positive and negative elliptic points of  $\mathcal{F}_{ob}(F)$  are equal;  $e_+(\mathcal{F}_{ob}) = e_-(\mathcal{F}_{ob})$ .

The first two properties imply that if  $\mathcal{F}_{ob}(F)$  is essential then the hypotheses of Theorem 5.3 are always satisfied. Thus, in order to estimate  $c(\phi, C)$  we only need to count the number of singular points of an essential open book foliation  $\mathcal{F}_{ob}(F)$ .

Now we prove Theorem 7.3. The strategy of proof is strongly motivated by braid foliation theory, and is similar to [22, Theorem 1.2]. In fact, Theorem 7.3 can be seen as a generalization of [22, Theorem 1.2]. By using the Euler characteristic of a surface and the region decomposition we find an elliptic point  $v$  such that the number of hyperbolic point around  $v$  is small. Lemma 5.1 gives the desired estimate of  $c(\phi, C)$ .

*Proof of Theorem 7.3.* We construct a cellular decomposition of the surface  $F$  by modifying the region decomposition of  $\mathcal{F}_{ob}(F)$ . To this end, we construct a singular foliation (but not an open book foliation),  $\mathcal{F}'$ , on  $F$  by replacing each cc-pants of  $\mathcal{F}_{ob}(F)$  with three bc-annuli as shown in Figure 15: Though  $\mathcal{F}'$  is not derived from the intersection

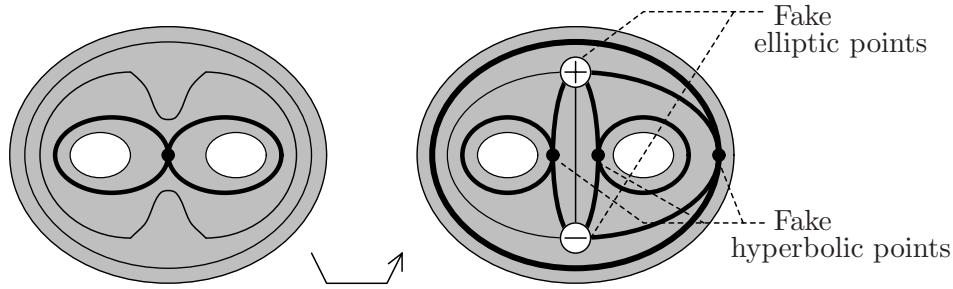


FIGURE 15. Construction of a singular foliation  $\mathcal{F}'$ .

of the surface  $F$  with the pages, by abuse of notations we keep using the terminologies of the open book foliation, such as region decomposition, bc-tile, bc-singular points, etc. We call the newly-inserted elliptic points and hyperbolic points *fake* elliptic points and *fake* hyperbolic points, respectively. The sign of a fake elliptic point is canonically determined. The sign of the hyperbolic points (both fake and non-fake) are not important in the following arguments so we omit it from now on.

The region decomposition of  $F$  induced by  $\mathcal{F}'$  consists only of bb-tiles and bc-annuli. Using Birman and Menasco's idea [9] we can construct a cellular decomposition of  $F$  as follows: A bc-annulus always exists in pair because the c-circle boundary is identified with the c-circle boundary of another bc-annulus. Let  $W$  be the annulus obtained by gluing two bc-annuli along their c-circles boundaries ( $W$  is a disc if one of the bc-annuli is degenerated). Each component of  $\partial W$  has two elliptic points. Choose two disjoint essential arcs each of which connects elliptic points of opposite sign as shown in Figure 16. Call such arcs *e-edges*. We cut  $W$  along the e-edges to obtain two 2-cells, which we name *be-tiles*.

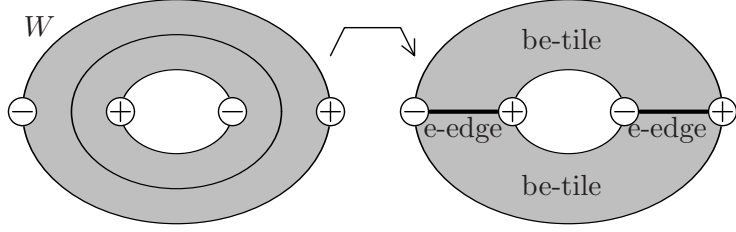


FIGURE 16. From bc-annuli to be-tiles. (cf. [9, Fig 18])

Now we view the surface  $F$  as a union of bb-tiles and be-tiles. This defines a cellular decomposition of  $F$  such that 0-cells (vertices) are the elliptic points, 1-cells are the b-arcs and e-edges on the boundary of the bb- and be-tiles, and 2-cells are the bb-tiles and be-tiles.

Let  $v$  be a 0-cell and  $\text{Val}(v)$  denote the valence of  $v$  in the 1-skeleton graph. We define  $\text{Hyp}(v)$  to be the number of non-fake hyperbolic points in  $\mathcal{F}_{ob}(F)$  that are connected to  $v$  by a singular leaf.

**Claim 7.6.** *let  $v$  be a 0-cell of the cellular decomposition of  $F$ .*

- (1) *If  $v$  is a fake elliptic point of  $\mathcal{F}'$ , then  $\text{Val}(v) = 6$ .*
- (2) *If  $v$  is not a fake elliptic point of  $\mathcal{F}'$ , then  $\text{Hyp}(v) \leq \text{Val}(v)$ .*

*Proof of claim 7.6.* (1) If  $v$  is a fake vertex there exists a cc-pants  $P$  in the region decomposition for  $\mathcal{F}_{ob}(F)$  on which  $v$  lies. In  $\mathcal{F}'$ ,  $P$  decomposes into three bc-annuli. After converting bc-annuli to be-tiles, at  $v$  three b-arc 1-cells and three e-edge 1-cells meet, see Figure 17. Totally  $\text{Val}(v) = 6$ .

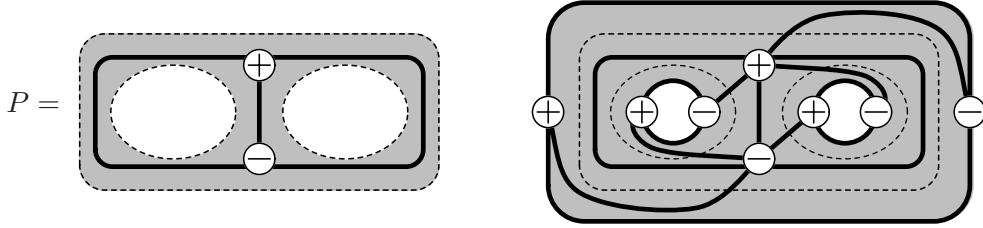


FIGURE 17. A fake elliptic point has valence six.

(2) We first note that a vertex  $v$  is a non-fake elliptic point in  $\mathcal{F}'$  if and only if it is an elliptic point of the original foliation  $\mathcal{F}_{ob}(F)$ . If  $v$  lies in the interior of a degenerated bc-annulus in  $\mathcal{F}_{ob}(F)$  then  $\text{Hyp}(v) = 1 < 2 = \text{Val}(v)$ .

Next let us assume that  $x$  bb-tiles and  $y$  bc-annuli in  $\mathcal{F}_{ob}(F)$  meet at  $v$ . Then  $\text{Hyp}(v) = x + y \leq x + 2y \leq \text{Val}(v)$  because each bb-tile contains one non-fake hyperbolic point and each bc-annulus contains one e-edge ending at  $v$ .  $\square$

Let us define

$$s = \min \{ \text{Hyp}(v) \mid v \text{ is a 0-cell and non-fake elliptic point} \}.$$

Fix a vertex  $v$  realizing  $\text{Hyp}(v) = s$ . Suppose that  $v$  lies on the binding component  $C \subset \partial S$ . Observation 7.5-(1) guarantees that  $v$  is strongly essential hence by Lemma 5.1

$$-s \leq c(\phi, C) \leq s.$$

Our goal is to show that  $s \leq 3$  if  $g = 0$  and  $s \leq 4 + \lfloor \frac{4g-4}{n} \rfloor$  if  $g > 0$ .

Consider the cellular decomposition of  $F$ . Let  $V(i)$  be the number of 0-cells of valence  $i$ ,  $E$  be the number of 1-cells, and  $R$  be the number of 2-cells, or the number of bb-tiles and be-tiles. Since each 1-cell is a common boundary of distinct two 2-cells and each 2-cell has distinct four 1-cells on its boundary, we have:

$$(7.1) \quad E = 2R.$$

Since the end points of a 1-cell are distinct two 0-cells we have:

$$(7.2) \quad \sum_i iV(i) = 2E.$$

The Euler characteristic of  $F$  is:

$$(7.3) \quad \sum_i V(i) - E + R = \chi(F)$$

From (7.1), (7.2) and (7.3), we get the *Euler characteristic equality*:

$$(7.4) \quad \sum_i (4 - i)V(i) = 4\chi(F).$$

First we assume that  $F$  is a sphere. The Euler characteristic equality (7.4) implies:

$$3V(1) + 2V(2) + V(3) = 8 + \sum_{i \geq 4} (i - 4)V(i)$$

The right hand side is positive, so there exists a vertex  $v$  with  $\text{Val}(v) \leq 3$ . By Claim 7.6-(1),  $v$  is not a fake vertex. By claim 7.6-(2) we obtain

$$s \leq \text{Hyp}(v) \leq \text{Val}(v) \leq 3.$$

This concludes (1).

Next we assume that  $F$  has genus  $g > 0$  so  $\chi(F) = 2 - 2g \leq 0$ . The Euler characteristic equality (7.4) is that:

$$3V(1) + 2V(2) + V(3) + 8g - 8 = \sum_{i \geq 4} (i - 4)V(i) \geq 0.$$

If at least one of  $V(1), V(2)$  and  $V(3)$  is positive then by Claim 7.6 there exists a non-fake elliptic point  $v$  such that  $s \leq \text{Hyp}(v) \leq \text{Val}(v) \leq 3$ . Suppose that  $V(1) = V(2) = V(3) = 0$ . By Observation 7.5-(3) the original open book foliation  $\mathcal{F}_{ob}(F)$  contains an even number  $(2n)$  of elliptic points. Therefore;

$$8g - 8 = \sum_{i \geq 4} (i - 4)V(i) \geq (s - 4)2n,$$

i.e.,  $s \leq 4 + \frac{4g-4}{n}$ . In either case since  $s$  is an integer  $s \leq 4 + \lfloor \frac{4g-4}{n} \rfloor$ . □

Finally we prove Theorem 7.4 by using Theorem 5.3 and Observation 7.5.

*Proof of Theorem 7.4.* We see in Observation 7.5-(3) that  $e_-(\mathcal{F}_{ob}(F)) = e_+(\mathcal{F}_{ob}(F)) = n$ . Let  $v_1^+, \dots, v_n^+$  be the positive elliptic points and  $v_1^-, \dots, v_n^-$  be the negative elliptic points of  $\mathcal{F}_{ob}(F)$ . By Proposition 2.5-(2),

$$(7.5) \quad h_+ + h_- = -\chi(F) + e_+ + e_- = 2g - 2 + 2n.$$

Since  $\partial S$  is connected, all the elliptic points lie on  $\partial S$ . So by Theorem 5.3 we have:

$$|c(\phi, \partial S)| \leq \min \left\{ \inf_{m \in \mathbb{N}} \frac{[f_+(m)]}{m}, \inf_{m \in \mathbb{N}} \frac{[f_-(m)]}{m} \right\}$$

where the functions  $f_{\pm}(m) : \mathbb{N} \rightarrow \mathbb{Q}$  satisfy:

$$f_{\pm}(m) \leq \begin{cases} \frac{h_{\pm}m}{n} - \frac{(n-1)^2}{4n^2} & (n : \text{odd}) \\ \frac{h_{\pm}m}{n} - \frac{n-2}{4n} & (n : \text{even}) \end{cases}$$

Hence by the equation (7.5) we get:

$$|c(\phi, \partial S)| \leq \begin{cases} \inf_{m \in \mathbb{N}} \left( \frac{1}{m} \left\lceil \frac{(g-1+n)m}{n} - \frac{(n-1)^2}{4n^2} \right\rceil \right) & (n : \text{odd}) \\ \inf_{m \in \mathbb{N}} \left( \frac{1}{m} \left\lceil \frac{(g-1+n)m}{n} - \frac{n-2}{4n} \right\rceil \right) & (n : \text{even}) \end{cases}$$

□

## 8. GEOMETRIC STRUCTURES OF OPEN BOOK MANIFOLDS

Finally we apply results in Section 7 to study geometric structures of open book manifolds. First we study Seifert-fibered case.

**Theorem 8.1.** *Assume that  $\phi \in \text{Aut}(S, \partial S)$  is periodic and  $c(\phi, C) \neq 0$  for every boundary component  $C$  of  $S$ . Then the 3-manifold  $M = M_{(S, \phi)}$  is Seifert-fibered.*

*Proof.* Let us put  $c(\phi, C_i) = \frac{p_i}{q_i}$  where  $(p_i, q_i)$  are coprime integers and  $p_i > 0$ . Let

$$A_i = C_i \times [1, 2] \cong \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$$

be an annular neighborhood of  $C_i$ , where  $C_i$  is identified with  $C_i \times \{1\}$ . Since  $c(\phi, C_i) = \frac{p_i}{q_i}$ , we may choose  $A_i$  so that:

- $\phi(A_i) = A_i$ .
- $\phi(z) = z \exp(-2\pi\sqrt{-1}(|z| - 1)\frac{p_i}{q_i})$  for  $z \in A_i = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ .

Let  $B_i$  be the binding component corresponding to  $C_i$ . Take a regular neighborhood  $N_i$  of  $B_i$  so that  $N_i \cap S_t \cong A_i$  for all  $t \in [0, 1]$ . The complement of the binding  $M \setminus \cup_i N_i \cong M_\phi$ , the mapping torus of  $\phi$ , is Seifert-fibered as  $\phi$  is periodic. The fibers on  $\partial N_i \subset \partial M_\phi$  are regular and each represents the homology class  $p_i[C_i] + q_i[S^1] \in H_1(\partial N_i; \mathbb{Z})$  where  $[S^1]$  corresponds to a meridian disc of  $N_i$ .

The assumption  $c(\phi, C_i) = \frac{p_i}{q_i} \neq 0$  implies that the Seifert fibration of  $M_\phi$  extends to  $N_i$ , by adding the binding  $B_i$  as an exceptional fiber of the Seifert invariant  $(\alpha_i, \beta_i)$  where  $0 \leq \beta_i < \alpha_i = p_i$  and  $\beta_i \equiv q_i \pmod{p_i}$ . □

**Remark.** It is necessary to assume  $c(\phi, C_i) \neq 0$ . For example, if  $S = S_{g,1}$  is a genus  $g$  surface with one boundary then  $M_{(S, id)} = \#_{2g}(S^1 \times S^2)$ , which admits no Seifert fibered structure as  $\mathbb{R}P^3 \# \mathbb{R}P^3$  is the only Seifert fibered manifold that is not prime (cf. [19]).

Next we observe the following rather simple fact. We adopt a convention that a lens space is a 3-manifold admitting a genus one Heegaard decomposition: so we will regard  $S^3$  and  $S^2 \times S^1$  as lens spaces.

**Lemma 8.2.** *Assume that  $\phi$  is periodic and that  $\partial S$  is connected. If  $|c(\phi, \partial S)| \neq 0, \frac{1}{N}$  for any  $N \in \mathbb{Z} - \{0\}$ , then  $M_{(S, \phi)}$  is not a lens space.*

*Proof.* By the hypothesis on  $c(\phi, \partial S)$ , we may put  $c(\phi, \partial S) = \frac{p}{q}$  where  $p > 1$  and  $p, q \in \mathbb{Z}$  are coprime. Since  $\partial S$  is connected and  $|c(\phi, \partial S)| \neq 0$ ,  $S$  is not a disc and  $\chi(S) < 0$ .

By Theorem 8.1 the manifold  $M$  admits a Seifert fibration such that the binding yields an exceptional fiber of Seifert invariant  $(p, q')$  where  $0 < q' < p$  and  $q' \equiv q \pmod{p}$ . Thus for some  $g, k \geq 0$  we have

$$M \simeq M(g, (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k), (p, q')).$$

That is, let  $\Sigma_\circ$  be a  $(k+1)$ -punctured genus  $g$  surface. Then  $M$  is the Seifert fibered manifold obtained from  $\Sigma_\circ \times S^1$  by filling  $(k+1)$  exceptional fibers of Seifert invariants  $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k), (p, q')$ . See [26] for notation. Topologically  $\Sigma_\circ$  is  $(S - A)/\phi$  with the boundary capped off by a disc (with a cone point), where  $A$  is the annular neighborhood of  $\partial S$  as in the proof of Theorem 8.1. The quotient map

$$\pi : (S - A) \rightarrow (S - A)/\phi$$

is a branched covering with  $k$  branch points, say  $p_1, \dots, p_k$ . The branch point  $p_i$  corresponds to the exceptional fiber of the ramification index  $\alpha_i$ . Thus,  $\alpha_i \geq 2$ .

If  $g \neq 0$  then  $M$  cannot be a lens space.

If  $g = 0$  then  $(S - A)/\phi$  is topologically a disc. Recall that we have seen that  $\chi(S) < 0$ , so  $\chi((S - A) \setminus \{p_1, \dots, p_k\}) \leq \chi(S) < 0$ . Away from the branch points  $\pi$  is a regular covering, so  $\chi(((S - A)/\phi) \setminus \{p_1, \dots, p_k\}) = 1 - k < 0$ , i.e.  $k \geq 2$ . Hence the Seifert fibration of  $M$  has at least three exceptional fibers of Seifert invariant  $(\alpha_i, \beta_i)$  and  $(p, q')$  with  $\alpha_i, p > 1$  and  $i = 1, 2$ . This shows that  $M$  is not a lens space.  $\square$

Next we switch to hyperbolic case. Recall that  $M_{(S, \phi)}$  is obtained from the mapping torus  $M_\phi$  by gluing solid tori along the boundary. In light of Thurston's hyperbolic Dehn surgery theorem, when  $\phi$  is pseudo-Anosov  $M_{(S, \phi)}$  is likely to admit a hyperbolic structure. However, there exist infinitely many pseudo-Anosov open books which yield non-hyperbolic 3-manifolds: Monodromies of hyperbolic fibered knots in  $S^3$  are such examples. With open book foliation technique, as a corollary of Theorem 7.2, we can give a simple criterion when a pseudo-Anosov open book gives rise to a hyperbolic 3-manifold.

**Corollary 8.3.** *Assume that:*

- (1)  $\phi$  is freely isotopic to an irreducible homeomorphism of  $S$ .
- (2)  $|c(\phi, C)| > 4$  for every boundary component  $C \subset \partial S$ , or  $\partial S$  is connected and  $|c(\phi, \partial S)| > 1$ .
- (3)  $\phi_* - id$  is not surjective.



Then  $M_{(S,\phi)}$  is hyperbolic.

*Proof.* By Theorem 7.2, conditions (1), (2) imply that  $M$  is irreducible and atoroidal.

For a relative 1-cycle  $c \in C_1(S, \partial S)$  notice that  $\phi_*c - c$  is a 1-cycle of  $C_1(S)$ , so we get a homomorphism:  $\phi_* - id : H_1(S, \partial S; \mathbb{Q}) \rightarrow H_1(S; \mathbb{Q})$ . By Etnyre and Ozbagci's explicit description of  $\pi_1(M)$  [15, p.3136] the first Betti number of  $M_{(S,\phi)}$  is given by the formula:

$$b_1(M_{(S,\phi)}) = \dim \text{Coker}(\phi_* - id) = \dim H_1(S; \mathbb{Q}) - \text{rank}(\phi_* - id).$$

Hence the condition (3) implies  $\pi_1(M)$  is infinite.

According to the geometrization theorem, a closed 3-manifold is hyperbolic if and only if it is irreducible, atoroidal, and has infinite fundamental group.  $\square$

**Corollary 8.4.** *If  $M$  is not a rational homology 3-sphere, then it is Haken so by Theorem 7.2 and the Thurston's hyperbolization imply that  $M$  is hyperbolic.*

If  $\partial S$  is connected, we have much tighter relationships among Nielsen-Thurston classification, fractional Dehn twist coefficients, and geometric structures. Recall that the mapping torus  $M_\phi$  of  $\phi \in \text{Aut}(S)$  is Seifert-fibered (toroidal, hyperbolic) if and only if  $\phi$  is periodic (reducible, pseudo-Anosov). In [22, Theorem 1.3] the first author generalized this to the complements of closed braids in  $S^3$  by using braid foliation technique. In the next theorem assuming that  $S$  has connected boundary we prove a parallel statement for  $M = M_{(S,\phi)}$ .

**Theorem 8.5.** *Let  $(S, \phi)$  be an open book decomposition of 3-manifold  $M$ . Assume that  $\partial S$  is connected, and  $|c(\phi, \partial S)| > 1$ .*

- (1)  *$M$  is toroidal if and only if  $\phi$  is reducible.*
- (2)  *$M$  is hyperbolic if and only if  $\phi$  is pseudo-Anosov.*
- (3)  *$M$  is atoroidal and Seifert fibered if and only if  $\phi$  is irreducible and periodic.*

*Proof.* ( $\Rightarrow$ ) of (1) is proved in Theorem 7.2.

( $\Leftarrow$ ) of (2): By a work of Roberts [33, Theorem 4.7], [21, Theorem 4.1], if  $\phi$  is pseudo-Anosov and  $|c(\phi, \partial S)| > 1$  then  $M$  admits a taut foliation, i.e.,  $\pi_1(M)$  is infinite. Theorem 7.2 shows that  $M$  is atoroidal and irreducible. Hence the hyperbolization theorem implies that  $M$  is hyperbolic.

( $\Leftarrow$ ) of (1): Assume that  $\phi$  is reducible. There exists an essential simple closed curve  $C$  in  $S$  such that  $\phi^n(C) = C$  for some  $n \in \mathbb{N}$ . Let  $C_i = \phi^{i-1}(C)$  where  $i = 1, \dots, n$ . We may assume that  $C_i$ 's are mutually disjoint. Let  $\mathcal{C} = C_1 \cup \dots \cup C_n$ . Then  $\mathcal{C} \times [0, 1] \subset S \times [0, 1]$  gives rise to an embedded torus  $\mathcal{T} = \mathcal{T}_{\mathcal{C}}$  in  $M = M_{(S,\phi)}$ . Our goal is to prove that  $C$  can be chosen so that  $\mathcal{T}$  is essential.

Assume contrary that  $\mathcal{T}$  is inessential. Compressing  $\mathcal{T}$  yields an embedded sphere in  $M$ . This sphere bounds a 3-ball in  $M$  as  $M$  is irreducible by Theorem 7.1-(2). Therefore  $\mathcal{T}$  bounds a solid torus  $X = X_C$ .

**Claim 8.6.** (i)  $\partial S \subset X \cap S_0$  and (ii)  $X \cap S_0$  is connected.

*Proof.* (i): If  $\partial S \not\subset X \cap S_0$  then  $\partial(X \cap S_0) = \mathcal{T} \cap S_0 = \mathcal{C}$ . So we obtain  $X \simeq S' \times S^1$  for some connected component  $S'$  of  $X \cap S_0$ . Since every  $C_i$  is essential,  $S'$  is not a disc, which contradicts the fact that  $X \simeq D^2 \times S^1$ .

(ii): If  $X \cap S_0$  has more than one component then the component of  $X \cap S_0$  containing  $\partial S$  is mapped to itself under  $\phi$  since  $\phi$  fixes  $\partial S$  point-wise. This means  $X$  has more than one component.  $\square$

Now we have  $\partial(X \cap S_0) = \partial S \cup C_1 \cup C_2 \cup \dots \cup C_n$ . We cap off  $X \cap S_0$  along  $C_1, \dots, C_n$  by discs  $D_1, \dots, D_n$  to obtain a connected surface  $\widehat{X \cap S_0}$  with connected boundary. We may denote the boundary of  $\widehat{X \cap S_0}$  by  $\partial S$ . Let  $\widehat{\phi}$  be the homeomorphism of  $\widehat{X \cap S_0}$  naturally extending  $\phi|_{X \cap S_0}$ . Consider the open book  $(\widehat{X \cap S_0}, \widehat{\phi})$  and denote  $\widehat{M} = M_{(\widehat{X \cap S_0}, \widehat{\phi})}$ .

With no loss of generality, we assume that the essential curve  $C$  is chosen so that  $X_C \cap S_0$  is *minimal*; that is, for another essential simple closed curve  $C' \subset S$  with  $\phi^{N'}(C') = C'$  for some  $N' > 0$  we have  $X_{C'} \cap S_0 \not\subset X_C \cap S_0$ .

**Claim 8.7.** *The minimality assumption on  $X_C \cap S_0$  implies that  $\widehat{\phi}$  is irreducible.*

*Proof.* Suppose that there exists an essential simple closed curve  $C' \subset \widehat{X_C \cap S_0}$  such that  $\widehat{\phi}^m(C') = C'$  for some  $m \in \mathbb{Z}$ . This induces a torus  $\mathcal{T}_{C'} \subset X_C \subset \widehat{M}$ . By the minimality assumption the torus  $\mathcal{T}_{C'}$  fails to bound a solid torus in  $\widehat{M}$ . Hence  $\mathcal{T}_{C'}$  is essential in  $X_C$ . This is impossible as  $X_C$  is a solid torus.  $\square$

The centers of the attached discs  $p_i \in D_i \subset \widehat{X \cap S_0}$  ( $i = 1, \dots, n$ ) give rise to an  $n$ -stranded closed braid  $\widehat{\beta}$  in the open book  $(\widehat{X \cap S_0}, \widehat{\phi})$ . Thus, attached discs  $\{D_i\}$  give rise to a solid torus  $N(\widehat{\beta})$ , a regular neighborhood of  $\widehat{\beta}$  in  $\widehat{M}$ . Note that  $\widehat{M} - N(\widehat{\beta}) \simeq X$  which is also a solid torus. Hence  $\widehat{M}$  is made of two solid tori, i.e.,  $\widehat{M}$  is a lens space.

**Claim 8.8.**  $\widehat{X \cap S_0} \neq D^2$ .

*Proof.* Assume that  $\widehat{X \cap S_0}$  is a disc then  $\widehat{M} \simeq S^3$ . Since  $X \simeq \widehat{M} - N(\widehat{\beta})$  and  $X$  is a solid torus, the closed braid  $\widehat{\beta}$  is the unknot in  $\widehat{M}$ .

We will show that this is impossible, by using the first author's work on the Dehornoy floor of braids and knots [23]. Let us regard  $\widehat{\beta}$  as the closure of an  $n$ -braid  $\beta$ , and identify  $\beta = \widehat{\phi} \in \mathcal{MCG}(\widehat{X \cap S_0} - \{p_1, \dots, p_n\})$  as an element of the mapping class group of the  $n$ -punctured disc. Let  $\gamma$  be an arc in  $\widehat{X \cap S_0}$  from a point on  $\partial S$  to  $p_1$ . By the assumption  $|c(\phi, \partial S)| > 1$ , we have

$$T_{\partial S}(\gamma) > \beta(\gamma) \quad \text{or} \quad \beta(\gamma) > T_{\partial S}^{-1}(\gamma).$$

Thus,  $\beta$  contains at least one positive or negative full-twist and  $[\beta]_D$ , the *Dehornoy floor* of  $\beta$  (an analogous notion in braid theory of the absolute value of the fractional Dehn twist coefficient), is not zero. In [23], it is shown that

$$[\beta]_D \leq g(\widehat{\beta})$$

where  $g(\widehat{\beta})$  is a genus of the knot  $\widehat{\beta}$ . Hence we conclude  $\widehat{\beta}$  is not the unknot.  $\square$

**Claim 8.9.**  $|c(\widehat{\phi}, \partial S)| > 1$ .

*Proof.* Let  $\gamma$  be an essential arc in  $\widehat{X \cap S_0}$ . Since  $\widehat{X \cap S_0} \neq D^2$  such an arc exists. Regarding  $\gamma$  as an arc in  $S$ , we observe:  $T_{\partial S}(\gamma) > \phi(\gamma)$  or  $\phi(\gamma) > T_{\partial S}^{-1}(\gamma)$ . Since  $\phi(\gamma) = \widehat{\phi}(\gamma)$  by Lemma 4.6,  $|c(\widehat{\phi}, \partial S)| > 1$ .  $\square$

Since  $\widehat{M}$  is a lens space, Lemma 8.2 implies that  $\widehat{\phi}$  is not periodic. As  $\widehat{\phi}$  is irreducible (Claim 8.7) it is of pseudo-Anosov. Then by  $(\Leftarrow)$  of (2),  $\widehat{M}$  is hyperbolic, which is a contradiction. Therefore,  $\mathcal{T}_C \subset M$  must be an essential torus, so  $M$  is toroidal.

$(\Rightarrow)$  of (2): We observe that  $\phi$  is non-periodic and irreducible by Theorem 8.1 and the assertion (1). This concludes  $\phi$  is of pseudo-Anosov.

$(\Leftarrow)$  of (3) is proved by Theorem 8.1 and (1).

$(\Rightarrow)$  of (3) follows from (1) and (2).  $\square$

By stabilization of a given open book sufficient times one can always make  $\partial S$  connected while preserving the topological type of the underlying 3-manifold  $M_{(S,\phi)}$ . However, as shown in the next proposition, stabilized open books have “small” fractional Dehn twist coefficients. This partially explains why we need an open book with “large”  $c(\phi, C)$  to extract properties of  $M_{(S,\phi)}$ .

**Proposition 8.10.** *If an open book  $(S, \phi)$  is a stabilization of an open book  $(S', \phi')$ , then there exists a component  $C$  of  $\partial S$  such that  $|c(\phi, C)| \leq 1$ . Moreover, if  $\partial S$  is connected then  $|c(\phi, \partial S)| \leq \frac{1}{2}$ .*

*Proof.* By definition of stabilization,  $\phi = \phi' \circ T_\gamma^{\pm 1}$ , where  $\gamma$  denote the core of the plumbed Hopf band. We may regard  $S' \subset S$  and  $\phi' \in \text{Aut}(S, \partial S)$ . In the following we suppose that  $\phi = \phi' \circ T_\gamma$  (similar arguments hold when  $\phi = \phi' \circ T_\gamma^{-1}$ ).

Let  $\delta$  be the co-core of the plumbed Hopf band, i.e., an essential arc in  $S$ , and let  $C$  be a component of  $\partial S$  that contains (at least) one of the endpoints of  $\gamma$ . Then

$$T_C^{-1}(T_\gamma^{-1}\delta) \geq \phi(T_\gamma^{-1}\delta) = \phi'(\delta) = \delta > T_C(T_\gamma^{-1}\delta)$$

hence by Lemma 4.6 we have  $|c(\phi, C)| \leq 1$ .

Next we assume that  $\partial S$  is connected, which is possible only if  $\partial S'$  has exactly two components, say  $C_1$  and  $C_2$ . Take an integer  $N \geq \max\{|c(\phi', C_1)|, |c(\phi', C_2)|\}$ . Viewing  $C_1$  and  $C_2$  as simple closed curves embedded in  $S$ , for any essential arc  $l \subset S$  we have

$$(8.1) \quad T_{\partial S}^{-1}(l) \geq T_{C_1}^{-N} T_{C_2}^{-N}(l) \geq \phi'(l) \geq T_{C_1}^N T_{C_2}^N(l) \geq T_{\partial S}(l).$$

Observe that

$$T_{\partial S}^{-1}(T_\gamma^{-1}\delta) \geq T_{\partial S}^{-1}(T_\gamma\delta) \stackrel{(8.1)}{\geq} \phi'(T_\gamma\delta) = \phi^2(T_\gamma^{-1}\delta)$$

and

$$\phi^2(T_\gamma^{-1}\delta) = \phi'(T_\gamma\delta) \geq T_{C_1}^N T_{C_2}^N(T_\gamma\delta) \geq T_{\partial S}(T_\gamma^{-1}\delta).$$

Figure 18 justifies the last inequality  $T_{C_1}^N T_{C_2}^N(T_\gamma\delta) \geq T_{\partial S}(T_\gamma^{-1}\delta)$ . By Lemma 4.6 and Proposition 4.4-(1) we conclude  $|c(\phi, \partial S)| \leq \frac{1}{2}$ .  $\square$

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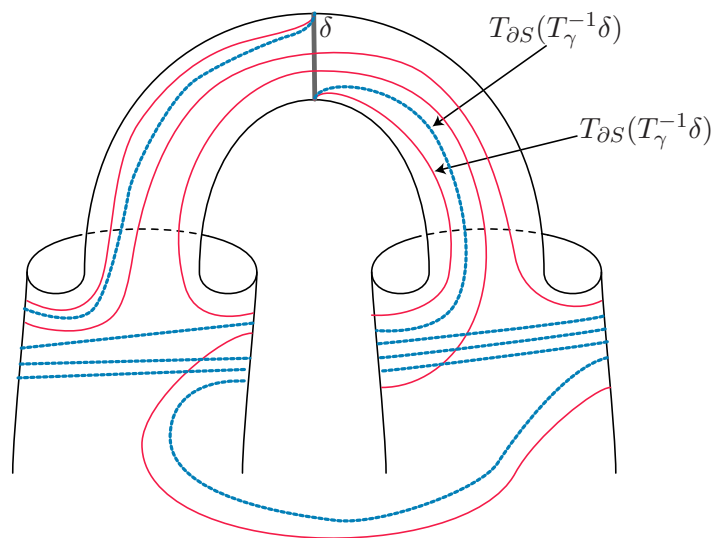


FIGURE 18.  $T_{C_1}^N T_{C_2}^N(T_{\gamma}\delta) \geq T_{\partial S}(T_{\gamma}^{-1}\delta)$  where  $N = 4$ .

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